Informally a smooth manifold is a space which locally looks like an open subset of $\mathbb{R}^n$. **It need not be a subset of $\mathbb{R}^N$ globally.**

Example 0.0.1. Let X be the configuration space of two distinct points in $\mathbb{R}^2$ i.e. the set of two point subsets of $\mathbb{R}^2$. It's equal to $(\mathbb{R}^2 \times \mathbb{R}^2 \backslash \Delta \mathbb{R}^2)/\sim$ where $(x, y) \sim (y, x)$. This space is a manifold but it's not canonically embedded into any $\mathbb{R}^N$.

Definition 0.0.2. *A* **generalized smooth manifold** *of dimension n is a set M together with a collection of subsets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and maps $\psi_\alpha \colon V_\alpha \to U_\alpha$ where V_α is an open subset of $\mathbb{R}^n$ such that the following properties are satisfied*

(1) $\cup_\alpha U_\alpha = M$
(2) $\psi_\alpha \colon V_\alpha \to U_\alpha$ is a bijection for every α
(3) For any α, β the set $U_{\alpha\beta} = \psi_\alpha^{-1}(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$
(4) For any α, β the map $\psi_\beta^{-1} \circ \varphi_\alpha \colon U_{\alpha\beta} \to U_{\beta\alpha}$ is smooth.

The collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ together with the maps $\psi_\alpha \colon V_\alpha \to U_\alpha$ is called a **smooth atlas** *on M.*

The maps $\psi_\alpha \colon V_\alpha \to U_\alpha$ are called **local parameterizations**.

The inverse maps $\varphi_\alpha = \psi_\alpha^{-1} \colon U_\alpha \to V_\alpha$ are called **local charts or local coordinates**.

Two atlases on M are said to define the same smooth structure if their union is still a smooth atlas on M.

Examples

- $M = \mathbb{R}^n, \mathcal{A} = \{1\}, U_1 = V_1 = \mathbb{R}^n, \psi_1 = id$.
- $M = \mathbb{R}, \mathcal{A} = \{1\}, U_1 = V_1 = \mathbb{R}, \psi_1 \colon \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$.
- $M = \Gamma_f$ - the graph of f where $f \colon \mathbb{R}^n \to \mathbb{R}^k$. (here $\Gamma_f = \{(x, y) \in \mathbb{R}^{n+k} | \quad y = f(x)\}$). $\mathcal{A} = \{1\}, U_1 = \mathbb{R}^n, V_1 = M, \psi(x) = (x, f(x))\}$
- $S^n = \{(x_0, x_1, \ldots, x_n) | \quad \Sigma_i x_i^2 = 1\}$. For each i set $U_i^+ = \{(x_0, x_1, \ldots, x_n) \in S^n | \quad x_i > 0\}$ and $U_i^- = \{(x_0, x_1, \ldots, x_n) \in S^n | \quad x_i < 0\}$.

  Let $V_i^\pm = \{(u_0, \ldots, u_{n-1}) | \quad \Sigma_i u_i^2 < 1\}$ and let $\psi_i^\pm \colon V_i^\pm \to U_i^\pm$ be given by $\psi_i(u) = (u_0, \ldots, u_{i-1}, \pm\sqrt{1 - |u|^2}, u_i, \ldots, u_{n-1})$. The inverse map is just the projection $\varphi_i^\pm(u_0, \ldots, u_n) = (u_0, \ldots, \hat{u}_i, \ldots u_n)$

  This collection is a smooth atlas on S^n (verify!).

  If $j < i$ then

$$\psi_j^{-1} \circ \psi_j(u_0, \ldots, u_{n-1}) = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{i-1}, \pm\sqrt{1 - |u|^2}, u_i, \ldots, u_{n-1})$$

- $\mathbb{RP}^n$ is the set of lines through 0 in $\mathbb{R}^{n+1}$. It can be equivalently described as the set of equivalence classes of points in $\mathbb{R}^{n+1} \backslash \{0\}$ modulo the equivalence relation $(x_0, \ldots, x_n) \sim (x_0', \ldots, x_n')$ iff $(x_0', \ldots, x_n') = t(x_0, \ldots, x_n)$ for some $t \neq 0$. The equivalence class of $(x_0, \ldots, x_n)$ will be denoted by $[x_0 : x_1 : \ldots : x_n]$

  Let $U_i \subset \mathbb{RP}^n$ be the set of lines of the form $\mathbb{R}(x_0, \ldots, x_n)$ with $x_i \neq 0$ for any nonzero point. any such line intersects the hyperplane

$\{x_i = 1\}$ in a single point. This gives us a bijective map $\varphi_i \colon U_i \to \mathbb{R}^n$ given by $\varphi_i([x_0 : x_1 : \ldots : x_n]) = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i})$ with the inverse given by

$\psi_i(u_0, \ldots, u_{n-1}) = [u_0 : \ldots : u_{i-1} : 1 : u_i : \ldots : u_n]$.

For $j < i$ we have $\varphi_j \circ \psi_i(u_0, \ldots, u_{n_1}) = \varphi_j([u_0 : \ldots, u_{i-1} : 1 : u_i : \ldots : u_{n-1}]) = (\frac{u_0}{u_j}, \ldots, \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, \ldots, \frac{u_{i-1}}{u_j}, \frac{1}{u_j}, \frac{u_{i-1}}{u_j}, \frac{u_i}{u_j}, \ldots, \frac{u_{n-1}}{u_j})$. This map is smooth on the open set $\{u_j \neq 0\} = \psi_i^{-1}(U_j)$. A similar formula holds in the case $j > i$. This gives a smooth atlas on $\mathbb{RP}^n$.

Example 0.0.3. Different smooth structures on $\mathbb{R}$. The first structure is given by $U_1 = V_1 = \mathbb{R}$ with $\psi \colon \mathbb{R} \to \mathbb{R}$ given by the identity map $\psi_1(x) = x$.

the second smooth structure is given by $\tilde{U}_1 = \tilde{V}_1 = \mathbb{R}$ with $\tilde{p}si \colon \mathbb{R} \to \mathbb{R}$ given by $\tilde{\psi}_1(x) = x^3$.

These smooth structure are distinct because these two atlases together do not form a smooth atlas since $\tilde{\psi}_1^{-1} \circ \psi_1(x) = \sqrt[3]{x}$ is not smooth at zero.

Definition 0.0.4. *let X be a generalized smooth n-dimensional manifold with an atlas $\{\psi_i \colon V_\alpha \to U_\alpha\}_{\alpha \in \mathcal{A}}$. A subset U of X is called* **open** *if $\psi_\alpha^{-1}(U)$ is an open subset of $\mathbb{R}^n$ for any α.*

It's easy to see that open sets satisfy the following properties

- X and $\emptyset$ are open.
- U_α is open for any α.
- Union of any collection of open sets is open
- Intersection of finitely many open sets is open.

Definition 0.0.5. *A generalized smooth manifold M^n is called Hausdorff for any two distinct points $p_1, p_2 \in X$ there exist open sets $W_1, W_2 \subset X$ such that $p_1 \in W_1, p_2 \in W_2$ and $W_1 \cap W_2 = \emptyset$.*

Definition 0.0.6. *A generalized smooth manifold M^n is called a smooth manifold if it is Hausdorff and it admits a* **countable** *atlas $\{\psi_i \colon V_\alpha \to U_\alpha\}_{\alpha \in \mathcal{A}}$.*

Example 0.0.7. Let $I_1 = (-1, 1) \times \{1\}, I_2 = (-1, 1) \times \{2\}$. Let X be the space obtained from $I_1 \cup I_2$ by identifying points $(x, 1)$ with $(x, 2)$ for all $x \neq 0$. Let $\pi \colon I_1 \cup I_2 \to X$ be the natural projection map and let $\psi_i \colon (-1, 1) \to X$ be given by $\psi_i(x) = \pi(x, i)$. This gives a smooth atlas on X with the transition map $\psi_2^{-1} \circ \psi_1$ equal to the identity map of $(-1, 1) \backslash \{0\}$ to itself. Thus X is a generalized 1−dimensional manifold. However it is not Hausdorff (why?) and hence is not a smooth manifold.

Example 0.0.8. $\mathbb{R}^n, S^n, \mathbb{RP}^n$ are Hausdorff and admit countable atlases. hence they are smooth manifolds.