## 1. Examples of smooth manifolds

Recall from the previous lecture that if $c$ is a regular value of $F\colon U \to \mathbb{R}^k$ where $U$ is an open subset of $\mathbb{R}^{n+k}$

then $M = \{F = c\}$ has a natural structure of a generalized manifold of dimension $n$.

**Example 1.0.1.** Let $SL(n,\mathbb{R})$ be the set of all $n \times n$ matrices with determinant 1. It is a smooth manifold of dimension $n^2 - 1$. To see this let $M(n \times m)$ be the set of all real $n \times m$ matrices. It can be canonically identified with $\mathbb{R}^{nm}$. $SL(n,\mathbb{R})$ is a subset of $M(n \times n)$ equal to the level set $\{F = 1\}$ of the function $F\colon M(n \times n) \to \mathbb{R}$ given by $F(A) = \det A$. Since the formula for the determinant of a matrix is polynomial in the coefficients of the matrix, it is obviously smooth.

We claim that 1 is a regular value of $F$. To see this we need to show that the differential $dF_A\colon \mathbb{R}^{n^2} \to \mathbb{R}$ is onto for any $A \in SL(n,\mathbb{R})$. Since the target is 1-dimensional this is equivalent to checking that $dF_A \neq 0$.

Let's find the formula for $dF_A$. We first consider the case $A = Id$.

**Claim:**

$$dF_{Id}(X) = tr(X)$$

for any $X \in M(n \times n)$.

Since both the left and the right side of this formula are linear in $X$ it's enough to verify it on the standard basis of $M(n \times n) = \mathbb{R}^{n^2}$.

Let $E_{ij}$ be the $n \times n$ matrix which has the $(i,j)$ entry equal to 1 and all other entries equal to 0.

Suppose $i \neq j$. Then $tr(E_{ij}) = 0$. On the other hand, $dF_{id}(E_{ij}) = D_{E_{ij}} F_{Id} = \lim_{t\to 0} \frac{F(Id+tE_{ij})-F(Id)}{t} = \lim_{t\to 0} \frac{1-1}{t} = 0$ since $Id + tE_{ij}$ is a triangular matrix with 1's on the diagonal.

Thus $dF_{Id}(E_{ij}) = tr(E_{ij})$ for any $i \neq j$.

Let's now consider the case $i = j$. Obviously, $tr(E_{ii}) = 1$.

As before we compute $dF_{id}(E_{ii}) = D_{E_{ii}} F_{Id} = \lim_{t\to 0} \frac{F(Id+tE_{ii})-F(Id)}{t} = \lim_{t\to 0} \frac{1+t-1}{t} = 1$ because $Id + tE_{ij}$ is a diagonal matrix with the $i$th diagonal entry equal to $1 + t$ and the rest of of diagonal elements equal to 1. $dF_{Id}(E_{ii}) = tr(E_{ii})$ for any $i$. Together with the above this means that $dF_{Id}(E_{ij}) = tr(E_{ij})$ for any $i, j$. By linearity of $tr$ and $dF_{Id}$ this proves the **Claim**.

Now suppose $A$ is an arbitrary matrix in $SL(n,\mathbb{R})$ and let $X \subset M(n\times n)$ be any matrix.

Then, using multiplicativity of determinants and the Claim above we compute

$$dF_A(X) = \lim_{t\to 0} \frac{F(A+tX) - F(A)}{t} = \lim_{t\to 0} \frac{\det(A+tX) - \det(A)}{t} =$$

$$= \lim_{t\to 0} \frac{\det(A(Id + tA^{-1}X) - \det(A)}{t} = \lim_{t\to 0} \frac{\det(A)\det(Id + tA^{-1}X) - \det(A)}{t} =$$

$$\det A \cdot \lim_{t\to 0} \frac{\det(Id + tA^{-1}X) - \det(Id)}{t} = tr(A^{-1}X)$$

Thus

$$dF_A(X) = tr(A^{-1}X)$$

This obviously means that $dF_A \neq 0$ (e.g. because $dF_A(A) = tr(A^{-1}A) = n$). Therefore 1 is a regular value of $F$ and hence $SL(n,\mathbb{R}) = \{F = 1\}$ is a smooth manifold of dimension $n^2 - 1$.

**Example 1.0.2.** Let $O(n) = \{A \in M(n\times n)|$ such that $A\cdot A^t = Id$. Then $O(n)$ is a smooth manifold. To see this, consider the map $F\colon M(n\times n) \to M(n\times n)$ given by $F(A) = A\cdot A^t$. Then $O(n) = \{F = Id\}$. However, $Id$ is not a regular value because $(A\cdot A^t)^t = (A^t)^t \cdot A^t = A\cdot A^t$ which means that $(A\cdot A^t)$ is symmetric for any $A$. Let $Sym(n)$ be the set of symmetric $n\times n$ matrices. Then $F$ maps $M(n\times n)$ to $Sym(n)$.

**Claim.** $Id$ is a regular value of $F\colon M(n\times n) \to Sym(n)$ Consequently, $O(n)$ is a smooth manifold of dimension $n(n-1)/2$. (Homework).

## 2. Manifolds with boundary

Let $\mathbb{H}^n = \mathbb{R}^n_+ = \{(x_1,\ldots,x_n) \in \mathbb{R}^n|$ such that $x_n \geqslant 0\}$. We will call the set $\{x_n > 0\}$ the *interior* of $\mathbb{H}^n$ and denote it by $\operatorname{int}\mathbb{H}^n$. We will call the set $\{x_n = 0\}$ the *boundary* of $\mathbb{H}^n$ and denote it by $\operatorname{int}\mathbb{H}^n$.

A subset $U \subset H^n$ is said to be open in $H^n$ if $U = W \cap H^n$ for some open set $W \subset \mathbb{R}^n$.

**Definition 2.0.3.** Let $U \subset \mathbb{H}^n$ be open. A map $F\colon U \to \mathbb{R}^k$ is called *smooth* if for every point $x \in U$ there exists an open set $W \subset \mathbb{R}^n$ containing $x$ such that $W \cap \mathbb{H}^n \subset U$ and $F|_{U\cap W}$ admits a smooth extension $\bar{F}\colon W \to \mathbb{R}^k$.

(Note that $\bar{F}$ *need not be unique*).

**Definition 2.0.4.** A *generalized smooth manifold with boundary* is a set $X$ together with a atlas $\{\psi_\alpha\colon V_\alpha \to U_\alpha \subset X\}_{\alpha\in A}$ where $V_\alpha$ is an open subset of $H^n$ such that the following properties are satisfied

1. $\cup_\alpha U_\alpha = M$
2. $\psi_\alpha\colon V_\alpha \to U_\alpha$ is a bijection for every $\alpha$
3. For any $\alpha, \beta$ the set $U_{\alpha\beta} = \psi_\alpha^{-1}(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$
4. For any $\alpha, \beta$ the map $\psi_\beta^{-1} \circ \varphi_\alpha\colon U_{\alpha\beta} \to U_{\beta\alpha}$ is smooth.

Similar to usual manifolds one can define equivalence of atlases and prove the existence of a unique maximal atlas containing a given atlas for (generalized) manifolds with boundary.

As for manifolds without boundary one can define the topology on a generalized manifold with boundary: A subset $U \subset X$ is called open if $U \cap \psi_\alpha^{-1}(U_\alpha)$ is open in $\mathbb{H}^n$.

**Definition 2.0.5.** A generalized smooth manifold $X$ with boundary is called *a smooth manifold with boundary* if it admits a countable atlas and is Hausdorff.

**Example 2.0.6.**

- $\mathbb{H}^n$ is a smooth manifold with boundary
- $\bar{D}^n = \{x \in \mathbb{R}^n |$ such that $|x| \leqslant 1$ is a smooth manifold with boundary
- A (generalized) smooth manifold is a (generalized) smooth manifold with boundary (why?) If $M^n$ is a smooth manifold of dimension $n$ and $N^m$ is a smooth manifold with boundary of dimension $m$ then $M \times N$ is a smooth manifold with boundary of dimension $n+m$.
- Let $U \subset \mathbb{R}^n$ be open and let $F\colon U \to \mathbb{R}$ be smooth. Suppose $c \in \mathbb{R}$ is a regular value of $f$. Then $\{F \leqslant c\}$ and $\{F \geqslant c\}$ are smooth manifolds with boundary of dimension $n$.

  *Proof.* We will construct an atlas on $M = \{F \geqslant c\}$ as follows. Let $F(p) > c$. Then because $F$ is continuous, there is an $\epsilon > 0$ such that $B_\epsilon(p) \subset \{F > c\}$. Pick a sufficiently large $d > 0$ such that $B_\varepsilon(p + (0, \dots, 0, d) \subset \mathbb{H}^n$. Set $V_p = B_\varepsilon(p + (0, \dots, 0, d)$ and define $\psi_p\colon V_p \to M$ by the formula $\psi_p(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n - d)$.

  Now suppose $F(p) = c$. Since $c$ is a regular value of $F$ we have that $(\frac{\partial F}{\partial x_1}(p), \dots, \frac{\partial F}{\partial x_n}(p)) \neq 0$. For simplicity let's assume $\frac{\partial F}{\partial x_n}(p) \neq 0$. As in the proof of the Implicit function theorem consider the map $\Phi\colon U \to \mathbb{R}^n$ defined by $\Phi(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, F(x_1, \dots, x_{n-1}, x_n) - c)$. We have that the matrix of partial derivatives $[d\Phi(p)]$ of $\Phi$ is an upper triangular matrix with 1's on the diagonal except for the last entry which is equal to $\frac{\partial F}{\partial x_n}(p)$. Therefore $\det[d\Phi(p)] = \frac{\partial F}{\partial x_n}(p) \neq 0$ and the Inverse Function Theorem is applicable. It says that for some small $\epsilon$ the map $\Phi$ bijectively maps $B_\varepsilon(p)$ to an open set $W$ in $\mathbb{R}^n$ and its inverse $\Psi\colon W \to B_\varepsilon(p)$ is also smooth. Note that by construction $\Phi(\{F \geqslant c\} \cap B_\varepsilon(p)) = W \cap \mathbb{H}^n$ and $\Phi(\{F = c\} \cap B_\varepsilon(p)) = W \cap \partial\mathbb{H}^n$. Let $\Psi_p = \Phi^{-1}\colon V_p = W \cap \mathbb{H}^n \to (\{F \geqslant c\} \cap B_\varepsilon(p)$. This will be our parameterization of $M$ near $p$

  It is now easy to see that the collection of all $\Psi_p$ over $p \in M$ gives a smooth atlas satisfying the definition of a generalized manifold with boundary.

  The fact that the resulting generalized manifold with boundary is Hausdorff and admits a countable atlas is proved in the same way as for regular level sets and is left to the reader as an exercise. □

**Definition 2.0.7.** Let $X$ be a (generalized) smooth manifold with boundary. A point $p \in X$ is called *interior* if for *some* chart $\psi_\alpha\colon V_\alpha \to U_\alpha \subset X$ we have that $p = \psi_\alpha(p_\alpha)$ for some $p_\alpha \in \operatorname{int} \mathbb{H}^n$.

A point $p$ is called a boundary point of $X$ is for *some* chart $\psi_\alpha\colon V_\alpha \to U_\alpha \subset X$ we have that $p = \psi_\alpha(p_\alpha)$ for some $p_\alpha \in \partial\mathbb{H}^n$.

**Theorem 2.0.8** (Topological invariance of the boundary)**.** *Let $X^n$ be a (generalized) smooth manifold with boundary of dimension $n$.*

1. *Every point $p$ of $X$ is either an interior point of $X$ or a boundary point of $X$* **but not both.**
2. *Let $\operatorname{int} X$ be the set of interior points of $X$ and let $\partial X$ be the set of boundary points of $X$. Then $\operatorname{int} X$ is a a (generalized) smooth manifold of dimension $n$ and $\partial X$ is a a (generalized) smooth manifold of dimension $n-1$.*

*Proof.* To see (1) suppose that $p$ is both an interior and a boundary point of $X$ at the same time. This means that there exists parameterizations $\psi_\alpha\colon V_\alpha \to U_\alpha \ni p$ and $\psi_\beta\colon V_\beta \to U_\beta \ni p$ such that $V_\alpha, V_\beta$ are open subsets of $\mathbb{H}^n$, $p = \psi_\alpha(p_\alpha) = \psi_\beta(p_\beta)$ and $p_\alpha \in \operatorname{int}\mathbb{H}^n = \{x \in \mathbb{R}^n |\quad x_n > 0\}$ and $p_\beta \in \partial\mathbb{H}^n = \{x \in \mathbb{R}^n |\quad x_n = 0\}$. By taking a small ball around $p_\alpha$ and reducing the domain of $\psi_\alpha$ we can assume that $V_\alpha$ is an open subset of $\mathbb{R}^n$.

Consider the transition maps $\psi_{\alpha\beta} = \psi_\beta^{-1} \circ \psi_\alpha\colon V_{\alpha\beta} \to V_{\beta\alpha}$ and $\psi_{\beta\alpha}\colon V_{\beta\alpha} \to V_{\alpha\beta}$.

They are inverse to each other and WLOG we can assume that $\psi_{\beta\alpha}$ admits a smooth extension $\bar{\psi}_{\beta\alpha}$ to a set $W_{\beta_\alpha}$ which is open in $\mathbb{R}^n$. Since $\bar{\psi}_{\beta\alpha} \circ \psi_{\alpha\beta} = \psi_{\beta\alpha} \circ \psi_{\alpha\beta} = Id$ by the chain rule we conclude that $d_{p_\alpha}\psi_{\alpha_\beta}\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

By the Inverse Function Theorem this means that there exist an open ball $B_\epsilon(p_\alpha) \subset V_{\alpha\beta}$ such that its image under $\psi_{\alpha_\beta}$ in an open subset of $\mathbb{R}^n$ containing $p_\beta$.

This is a contradiction because the image of $V_{\alpha\beta}$ under $\psi_{\alpha_\beta}$ is equal to $V_{\beta\alpha}$ which is a subset of $\mathbb{H}^n$ and no ball around $p_\beta$ is contained in $\mathbb{H}^n$.

Part (2) follows easily from part (1) (Exercise) □

**Remark 2.0.9.** A smooth manifold with boundary $X$ is as smooth manifold (without boundary) if and only if $\partial X = \emptyset$.