1. Smooth maps

Definition 1.0.1. Let M^n, N^m be smooth manifolds, possibly with boundary. A map $f\colon M \to N$ is called *smooth* if for any $p \in M$ and any parameterization $\varphi\colon V' \to U'$ where U' is open in N and contains $f(p)$ there exists a local parameterization on M $\psi\colon V \to U$ where U is open in M and $p \in U$ such that

(1) $f(U) \subset U'$ and
(2) $\varphi^{-1} \circ f \circ \psi\colon V \to \mathbb{R}^m$ is smooth.

Smooth maps from M to $\mathbb{R}$ are called *smooth functions* on M.

Lemma 1.0.2. *Let $f\colon M \to N$ be a smooth map. Then f is continuous.*

Proof.
Let $W \subset N$ be open. We need to show that $f^{-1}(W)$ is open in M. Let $p \in f^{-1}(W)$, i.e. $f(p) \in W$. Pick a parameterization $\varphi\colon V'_p \to U'_p$ on N such that $f(p) \in U'_p \subset W$. By definition of smoothness there exists a local parameterization on M $\psi\colon V_p \to U_p$ such that $p \in U_p$ and $f(U_p) \subset U'_p \subset W$. Then $U_p \subset f^{-1}(W)$ and since U_p is open in M we have that $f^{-1}(W) = \cup_{p \in f^{-1}(W)} U_p$ is open. □

The following criterion shows that it's enough to check smoothness with respect to given atlases on M and N

Proposition 1.0.3 (Criterion of smoothness)**.** *Let M^n, N^m be manifolds, possibly with boundary, and let $\{\psi_\alpha\colon V_\alpha \to U_\alpha\}_{\alpha \in A}$ be an atlas on M and $\{\varphi'_\beta\colon V'_\beta \to U'_\beta\}_{\beta \in B}$ be an atlas on N.*

then $f\colon M \to N$ is smooth if and only if for any α, β the set $U_\alpha \cap f^{-1}(U'_\beta)$ is open in M and the map $(\varphi'_\beta)^{-1} \circ f \circ \psi_\alpha\colon (f \circ \psi_\alpha)^{-1}(V'_\beta) \to V'_\beta \subset \mathbb{R}^m$ is smooth.

Proof. Suppose the map f satisfies the conditions of the proposition we need to show that f is smooth. Let's first check that f is continuous.

Let $U' \subset N$ be open. We want to show that that $f^{-1}(U')$ is open in M. Since $U = \cup_\beta (U' \cap U'_\beta)$ it's enough to show that $f^{-1}(U' \cap U'_\beta)$ is open for every β.

Thus, WLOG we can assume that $U' \subset U'_\beta$ for some β'.

We are given that U' is open in N. therefore $(\varphi'_\beta)^{-1}(U')$ is open in V'_β.

Likewise, since $U_\alpha \cap f^{-1}(U'_\beta)$ is open in M, the set $W_{\alpha\beta} = \psi_\alpha^{-1}[U_\alpha \cap f^{-1}(U'_\beta)] = (f \circ \psi_\alpha)^{-1}(V'_\beta)$ is open in V_α.

We are given that the map $(\varphi'_\beta)^{-1} \circ f \circ \psi_\alpha\colon W_{\alpha\beta} \to V'_\beta \subset \mathbb{R}^m$ is smooth. In particular, it's continuous and therefore $((\varphi'_\beta)^{-1} \circ f \circ \psi_\alpha)^{-1}(\varphi'_\beta)^{-1}(U') = (f \circ \psi_\alpha)^{-1}(U')$ is open in V_α. This holds for every α and hence $f^{-1}(U')$ is open in M.

This proves that f is continuous.

We are now ready to check the definition of smoothness of f. Let $p \in M$ and let $\varphi\colon V' \to U'$ be a parameterization on N where U' is open in N and contains $f(p)$. let $\psi_\beta\colon V'_\beta \to U'_\beta$ be some parameterization in our chosen atlas on N such that $f(p) \in U'_\beta$.

Since f is continuous the set $f^{-1}(U' \cap U'_\beta)$ is open in M. let $\psi_\alpha\colon V_\alpha \to U_\alpha$ be a local parameterization on M such that $p \in U_\alpha$. By changing U_α to $U_\alpha \cap f^{-1}(U' \cap U'_\beta)$ we can assume that $U_\alpha \subset f^{-1}(U' \cap U'_\beta)$

We claim that this $\psi = \psi_\alpha$ satisfies all the properties in the definition of a smooth map. Condition (1) is now obvious. To check condition (2) we first notice that the map $(\varphi'_\beta)^{-1} \circ f \circ \psi_\alpha$ is smooth by the assumption of the proposition.

Therefore the map $\varphi^{-1} \circ f \circ \psi_\alpha = (\varphi^{-1} \circ \varphi'_\beta) \circ ((\varphi'_\beta)^{-1} \circ f \circ \psi_\alpha)$ is also smooth as a composition of smooth maps.

This proves that f is smooth. The only if implication of the proposition is left to the reader as an exercise. □

The following properties of smooth maps easily follow from the definition

- A constant map $f\colon M \to N$ is smooth
- The identity map $id\colon M \to M$ is smooth
- Composition of smooth maps is smooth, that is if $f\colon M \to N$ and $g\colon N \to P$ are smooth then $g \circ f\colon M \to P$ is smooth
- Let $f\colon M \to N$ be smooth and let $U \subset M$ be open. Then $f|_U\colon U \to N$ is smooth.
- $f = (f_1, f_2)\colon M \to N_1 \times N_2$ is smooth if and only if $f_i\colon M \to N_i$ is smooth for $i = 1, 2$.
- Let $f, g\colon M \to R$ be smooth functions. Then $f \pm g, f \cdot g$ are smooth and $\frac{f}{g}$ is smooth on $U = \{g \neq 0\}$
- (smoothness is a local) Let $\{U\alpha\}_{\alpha \in A}$ be an open cover of M. Then $f\colon M \to N$ is smooth if and only if $f|_{U_\alpha}\colon U_\alpha \to N$ is smooth for every α.

If M is a regular level set then smoothness of maps from M can be easily checked using the following proposition

Proposition 1.0.4. *Let $U \subset \mathbb{R}^{n+k}$ be open and let $F\colon U \to \mathbb{R}^k$ be s smooth map and $c \in \mathbb{R}^k$ be a regular value of F. Let $M^n = \{F = c\}$. Then the inclusion $i\colon M \to \mathbb{R}^{n+k}$ is a smooth map.*

Proof. This is immediate from the definition of the smooth structure on M and Proposition 1.0.3:

Recall that a smooth atlas on M is defined as follows. Given a point $p \in M$ after possibly renumbering coordinates in $\mathbb{R}^{n+k}$ we can assume that $F(x, y)$ satisfies $\det[\frac{\partial F_i}{\partial y_j}(p)] \neq 0$. By the inverse function theorem new p M is a graph of a smooth function $y = y(x)$ and we have a local parameterization ψ_p of M near p given by $\psi_p(x) = (x, y(x))$. The collection of all such ψ_p over $p \in M$ is an atlas on M. For an atlas on $\mathbb{R}^{n+k}$ we just take the identity map $id\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$. It is now immediate that the conditions of Proposition 1.0.3 hold for these atlases and hence $i\colon M \to \mathbb{R}^{n+k}$ is a smooth map. □

Corollary 1.0.5. *Under the assumptions of Proposition 1.0.4 suppose $g\colon U \to N$ is smooth. Then $g|_M\colon M \to N$ is also smooth.*

Proof. $g|_M = g \circ i$ where $i\colon M \to \mathbb{R}^{n+k}$ is the canonical inclusion. Hence $g|_M$ is smooth as a composition of smooth maps. □

2. Diffeomorphisms

Definition 2.0.6. A map $f\colon M \to N$ is called a *diffeomnorphism* if f is a bijection and both f and f^{-1} are smooth.

Two manifolds M and N are called diffeomorphic if there exists a diffeomorphism $f\colon M \to N$

Example 2.0.7.

(1) let $M = \mathbb{R}$. Then $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is not a diffeomorphism. It's 1-1, onto an d smooth but the inverse map $f^{-1}(y) = \sqrt[3]{y}$ is not smooth.
(2) Any two intervals (a, b) and (c, d) are diffeomorphic. For example $f(x) = \frac{d-c}{b-a}(x - a) + c$ is a diffeomorphism from (a, b) to (c, d)
(3) $f(x) = e^x$ gives a diffeomorphism from $\mathbb{R}$ to $(0, \infty)$.
(4) $f(x) = \tan x$ gives a diffeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$.

the following properties of diffeomorphisms easily follow from the definition

- Let $\{\psi_\alpha\colon V_\alpha \to U_\alpha\}_{\alpha\in A}$ be an atlas on M. Then $\psi_\alpha\colon V_\alpha \to U_\alpha$ is a diffeomorphism for any α.
- if $f\colon M \to N$ is a diffeomorphism then $f^{-1}\colon N \to M$ is also a diffeomorphism
- The identity map $id\colon M \to M$ is a diffeomorphism
- if $f\colon M \to N$ and $g\colon N \to P$ are diffeomorphisms then $g \circ f\colon M \to P$ is also a diffeomorphism.
- If $f\colon M \to N$ is a diffeomorphism and $U \subset M$ is open then $f|_U\colon U \to f(U)$ is also a diffeomorphism
- If $f\colon M \to N$ is a diffeomorphism then M is connected if and only if N is connected
- If $f\colon M \to N$ is a diffeomorphism then M is compact if and only if N is compact

Theorem 2.0.8 (Diffeomorphism invariance of boundary)**.** *Let $f\colon M \to N$ be a diffeomorphism. Then $f(\partial M) = \partial N$ and $f|_{\partial M}\colon \partial M \to \partial N$ is a diffeomorphism.*

Proof. The proof is exactly the same as showing that a point in manifold with boundary can not be an interior and boundary point at the same time. The details are left to the reader as an exercise. □

Example 2.0.9. The closed Möbius band M and the cylinder $N = S^1 \times [01]$ are not diffeomorphic because $\partial M = S^1$ and $\partial N = S^1 \times \{0\} \cup S^1 \times \{1\}$. Since ∂M is connected and ∂N is not connected ∂M is not diffeomorphic to ∂N and hence M is not diffeomorphic to N.