PRACTICE PROBLEMS ON INTEGRATION AND STOKES FORMULA

(1) Let M^n be a compact orientable manifold without boundary.
  (a) Let $\omega \in \Omega^n(M)$ be an exact n-form. Prove that there is a point $x \in M$ such that $\omega(x) = 0$.
  (b) Prove that $H^n_{DR}(M) \neq 0$.

(2) Let M^n be a smooth manifold, let T^*M be its cotangent bundle and let $\pi\colon T^*M \to M$ be the canonical projection map. Define a 1-form α on T^*M as follows. Let $p \in M, \eta \in T^*_p(M)$. Look at $d\pi\colon T_{(p,\eta)}T^*M \to T_pM$. Set

$$\alpha(p,\eta) = d\pi^*(\eta)$$

  (a) let $x\colon U \to V \subset \mathbb{R}^n$ be a local coordinate chart on M and let $(x,y) \mapsto \sum_{i=1}^n y^i dx^i$ be the corresponding local coordinates on $T^*(U)$
  Prove that with respect to these local coordinates α has the form

$$\alpha(x,y) = \sum_{i=1}^n y^i dx^i$$

  (b) Let $\omega = d\alpha$. Prove that $\underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}}$ is a nowhere zero $2n$-form on T^*M. (This in particular implies that T^*M is orientable).

(3) Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $U = R^3\backslash(0,0,0)$.
  (a) Prove that $d\omega = 0$.
  Hint: One way to simplify the computation is to write $\omega = f\cdot\tilde{\omega}$ where $f = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ and $\tilde{\omega} = xdy\wedge dz + ydz\wedge dx + zdx$. Another way is to use spherical coordinates on $\mathbb{R}^3\backslash\{0\}$.
  (b) Let $S^2 = \{(x,y,z) \in R^3|$ such that $x^2+y^2+z^2 = 1\}$ with the orientation induced from $B^3 = \{(x,y,z) \in R^3|$ such that $x^2+y^2+z^2 \leq 1\}$. Let $i\colon S^2 \to \mathbb{R}^3$ be the canonical inclusion. Show that $i^*(\omega)$ is nowhere zero on S^2.
  (c) Show that ω is not exact on U.

(4) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^{2y}, 2x+y))$ and let $\omega = x^2ydx + ydy$.
  Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

(5) Let $M^2 \subset \mathbb{R}^3$ be the torus of revolution obtained by rotating the circle $(x-2)^2 + z^2 = 1$ in the xz plane around the yz axis.
  (a) Prove that M^2 is orientable

   (b) Consider the orientation on M induced by the normal field N where $N(3,0,0) = (1,0,0)$.
   Find $\int_M x dy \wedge dz$.

(6) Let $M^3 = \{(x,y,z) \in \mathbb{R}^3|$ such that $6 \leq 2x^2 + y^2 + 3z^2 \leq 7\}$ with the orientation induced from R^3.
   (a) Show that M is a manifold with boundary.
   (b) Let $p = (1,1,1)$. Check that $p \in \partial M$ and find a positive basis of $T_p\partial M$ with respect to the orientation of ∂M induced from M.

(7) Let $a, b > 0$ and Let $M \subset \mathbb{R}^2$ be the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ with the orientation induced by the standard orientation on $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$.
   Find $\int_M (\cos x) y dx + (x + \sin(x)) dy$.