(1) Let M, N be smooth manifolds and let $f\colon M \to N$ be smooth. Prove that the following are equivalent
   (a) f is a smooth embedding;
   (b) $S = f(M)$ is a smooth submanifold of N and $f\colon M \to S$ is a diffeomorphism.

(2) Recall that given a smooth manifold with boundary M^n we call a set $S \subset M$ a submanifold (with boundary) of M of dimension k if for every $p \in S$ there exist an open set U containing p and a diffeomorphism $x\colon U \to V$ where V is an open set in H^n such that $x(U \cap S) = V \cap H^k$ for some $H^k \subset H^n$.

   Let $M = H^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and let $S = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$. Prove that S is not a submanifold in H^2 but it is a submanifold in $\mathbb{R}^2$.

(3) Let M be a manifold with boundary and let N be a manifold without boundary. Let $f\colon M \to N$ be smooth and let $c \in N$ be a regular value for both f and $f|_{\partial M}$.

   Prove that $S = f^{-1}(c)$ is a smooth submanifold with boundary in M and $\partial S \subset \partial M$.

(4) Define a map $f\colon \mathbb{S}^2 \to \mathbb{R}^4$ given by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that F induces a well defined smooth map $\bar{F}\colon \mathbb{RP}^2 \to \mathbb{R}^4$ and prove that $\bar{F}$ is a smooth embedding.

(5) Let $U, V \subset H^n$ be open sets containing $p = 0$ and let $f\colon U \to V$ be a diffeomorphism such that $f(p) = p$. Let $v = (v_1, \ldots, v_n)$ be a vector in $\mathbb{R}^n$ with $v_n > 0$.

   Let $df_p(v) = w = (w_1, \ldots, w_n)$.

   Prove that $w_n > 0$.

(6) Prove that $\mathbb{R}^n \backslash \{0\}$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$.