(1) Let V be a smooth vector field on a manifold M.

Prove that $S = V(M) \subset TM$ is a smooth submanifold and that it's closed as a subset of TM.

(2) Let $V = (V_1, \dots, V_n)$ be a vector field on an open set $U \subset \mathbb{R}^n$.

Prove that V is transverse to the zero section iff for any $p \in U$ such that $V(p) = 0$ we have

$$\det\left(\frac{\partial V_i}{\partial x_j}(p)\right) \neq 0$$

(3) Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Let $V(x, y, z) = (-y, x, 0)$.

Show that V is a smooth tangent vector field on S^2 and that it's transverse to the zero section.

(4) Let $f\colon M \to \mathbb{R}$ be a smooth function without critical points.

A vector field V on M is called *gradient-like* for f if $Vf > 0$ on M.

Prove that there exists a smooth gradient-like vector filed V.

Hint: Use partition of unity.

(5) Let $S \subset M$ be a submanifold. Let V be a smooth vector field on S.

Prove that there exists an open set U containing S and a smooth vector field $\tilde{V}$ on U such that $\tilde{V}|_S = V$.

Hint: Use partition of unity.

(6) Show that there exists a smooth vector field on S^1 with exactly one zero. Prove that any such vector field can not be transverse to the zero section.