(1) Let V be a smooth vector field on a smooth manifold M without boundary. Let $p \in M$ and suppose $V(p) \neq 0$.

Prove that there exist local coordinates x on an open set U containing p such that in these coordinates $V \equiv \frac{\partial}{\partial x_1}$.

Hint: Use an argument similar to the proof of the collar neighbourhood theorem.

(2) Let $\sigma \in S_{k+l}$ be the following permutation

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & k+l \\ l+1 & \cdots & l+k & 1 & \cdots & l \end{pmatrix}$$

Show that $\operatorname{sign}(\sigma) = (-1)^{kl}$

(3) Let V be a vector space of dimension n. An alternating k-tensor ω on V is called *decomposable* if it can be written as $\omega = \eta \wedge \nu$ where η and ν have degrees smaller than ω.

(a) Let $V = \mathbb{R}^4$ and $\omega = e^{12} + e^{34}$. Prove that ω is not decomposable.

(b) Let $V = \mathbb{R}^n$ where $n \geq 4$ and $\omega = e^{12} + e^{34}$. Prove that ω is not decomposable.

Hint: Given $\omega \in \Lambda^{n-2}(\mathbb{R}^n)$ consider the map $L_\omega \colon \Lambda^1(\mathbb{R}^n) \to \Lambda^{n-1}(\mathbb{R}^n)$ given by $L_\omega(\eta) = \omega \wedge \eta$. Look at the dimension of $\ker L_\omega$.

(4) Let V, W be finite-dimensional vector spaces and let $f \colon V \to W$ be a linear map. Using the definition of wedge product given in class prove that $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ for any alternating tensors ω, η on W.

(5) Let M^n be a smooth manifold. Let $\pi \colon \Lambda^k(M) =\to M$ be the canonical projection. Let $\omega \colon M \to \Lambda^k(M)$ be a section of π, i.e. a map satisfying $\pi \circ \omega = id_M$. Such ω is called a differential form on M. Prove that the following conditions are equivalent

(a) ω is a smooth map

(b) in any local coordinate chart $x \colon U \to V$ where $U \subset M, V \subset \mathbb{R}^n$ are open when we write ω as $\omega = \sum_I \omega_I(x) dx^I$ then all $\omega^I(x)$ are smooth functions.

(c) for any smooth vector fields $V_1, \dots V_k$ on M the function $f \colon M \to \mathbb{R}$ given by $f(p) = \omega(p)(V_1(p), \dots V_k(p))$ is smooth.

(6) Let $U \subset \mathbb{R}^n$ be open and let $V_1, \dots V_n$ be smooth vector fields on U such that for any $x \in U$, $V_1(x), \dots V_n(x)$ is a basis of $\mathbb{R}^n$.

Let $w_1, \dots w_n$ be the dual collection of 1 forms, i.e. for any $x \in U$ $w_1(x), \dots w_n(x)$ is the unique n-tuple of elements of $(\mathbb{R}^n)^*$ satisfying $w_i(x)(V_j(x)) = \delta_{ij}$.

Prove that all w_i are smooth.