(1) Let $V = \mathbb{R}^4$ with the standard basis $e_1, \dots, e_4$. Let $\omega = e^{1,3,4}$. Express ω as a linear combination of $\{\phi^I\}_{I=(i_1,i_2,i_3)}$ - the canonical basis of all 3-tensors o $\mathbb{R}^4$. Recall that for $I = (i_1, i_2, i_3)$ $\phi^I = e^{i_1} \otimes e^{i_2} \otimes e^{i_3}$.

(2) Prove that $H^1_{DR}(\mathbb{R}^2) = 0$.

Hint: Let $\omega = a(x,y)dx + b(x,y)dy$ be a closed 1-form on $\mathbb{R}^2$. Consider $f(x,y) = \int_0^x a(s,0)ds + \int_0^y b(x,t)dt$.

(3) A Riemannian metric on a manifold M^n is a smooth section $\langle \cdot, \cdot \rangle$ of the bundle of 2-tensors on M such that for any $p \in M$, $\langle \cdot, \cdot \rangle_p \colon T_pM \times T_pM \to \mathbb{R}$ is an inner product on T_pM.

   (a) Prove that any smooth manifold admits a Riemannian metric.

   (b) Two vector bundles $E_1 \xrightarrow{\pi_1} M$, $E_2 \xrightarrow{\pi_2} M$ are called *isomorphic* if there exists a diffeomorphism $F\colon E_1 \to E_2$ such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[rr, "F"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

   and $F\colon \pi_1^{-1}(x) \to \pi_2^{-1}(x)$ is a linear isomorphism for every $x \in M$.

   Prove that for any smooth manifold M it holds that TM is isomorphic to T^*M.

   Hint: use part a).