PRACTICE PROBLEMS ON INTEGRATION, STOKES FORMULA, DE RHAM COHOMOLOGY, MAYER-VIETORIS, DEGREE THEORY

(1) Let M^n be a compact orientable manifold without boundary.
Let $\omega \in \Omega^n(M)$ be an exact n-form. Prove that there is a point $x \in M$ such that $\omega(x) = 0$.

(2) Prove that $H^2(\mathbb{RP}^2) = 0$.

(3) Let M^n be a smooth manifold, let T^*M be its cotangent bundle and let $\pi\colon T^*M \to M$ be the canonical projection map. Define a 1-form α on T^*M as follows. Let $p \in M, \eta \in T_p^*(M)$. Look at $d\pi\colon T_{(p,\eta)}T^*M \to T_pM$. Set

$$\alpha(p,\eta) = d\pi^*(\eta)$$

(a) let $x\colon U \to V \subset \mathbb{R}^n$ be a local coordinate chart on M and let $(x,y) \mapsto \sum_{i=1}^n y^i dx^i$ be the corresponding local coordinates on $T^*(U)$
Prove that with respect to these local coordinates α has the form

$$\alpha(x,y) = \sum_{i=1}^n y^i dx^i$$

(b) Let $\omega = d\alpha$. Prove that $\underbrace{\omega \wedge \ldots \wedge \omega}_{n \quad \text{times}}$ is a nowhere zero $2n$-form on T^*M. (This in particular implies that T^*M is orientable).

(4) Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $U = R^3\backslash(0,0,0)$.

(a) Prove that $d\omega = 0$.
Hint: One way to simplify the computation is to write $\omega = f \cdot \tilde{\omega}$ where $f = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ and $\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx$.
Another way is to use spherical coordinates on $\mathbb{R}^3\backslash\{0\}$.

(b) Let $S^2 = \{(x,y,z) \in R^3|$ such that $x^2+y^2+z^2 = 1\}$ with the orientation induced from $B^3 = \{(x,y,z) \in R^3|$ such that $x^2+y^2+z^2 \le 1\}$. Let $i\colon S^2 \to \mathbb{R}^3$ be the canonical inclusion. Show that $i^*(\omega)$ is nowhere zero on S^2.

(c) Show that ω is not exact on U.

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^{2y}, 2x+y))$ and let $\omega = x^2ydx + ydy$.
Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

(6) Let $M^2 \subset \mathbb{R}^3$ be the torus of revolution obtained by rotating the circle $(x-2)^2 + z^2 = 1$ in the xz plane around the yz axis.

(a) Prove that M^2 is orientable
(b) Consider the orientation on M induced by the normal field N where $N(3,0,0) = (1,0,0)$.
Find $\int_M x dy \wedge dz$.

(7) Let $M^3 = \{(x,y,z) \in \mathbb{R}^3 |$ such that $6 \leq 2x^2 + y^2 + 3z^2 \leq 7\}$ with the orientation induced from R^3.
(a) Show that M is a manifold with boundary.
(b) Let $p = (1,1,1)$. Check that $p \in \partial M$ and find a positive basis of $T_p \partial M$ with respect to the orientation of ∂M induced from M.

(8) Let $a, b > 0$ and Let $M \subset \mathbb{R}^2$ be the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ with the orientation induced by the standard orientation on $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$.
Find $\int_M (\cos x) y dx + (x + \sin(x)) dy$.

(9) (a) Let $f, g \colon \mathbb{S}^n \to \mathbb{S}^n$ be continuous such that $f(x) \neq g(x)$ for any x.
Prove that $f \sim -g$.
(b) Let $f \colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ be continuous.
Prove that either f has a fixed point or there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = -x$
(c) Prove that any continuous map $f \colon \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

(10) Verify that the Euler characteristic $\chi(\mathbb{S}^2)$ is 2 by looking at the vector field $V(x,y,z) = (-y,x,0)$ where we view $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$.

(11) Let M^{2n+1} be a closed orientable manifold. Prove that $\chi(M) = 0$.