(1) Prove that the set of finite subsets of $\mathbb{N}$ is countable.

Solution

Let S_k be the set of subsets of N consisting of k elements. Then $S = \cup_{k=1}^{\infty} S_k$. Let $f_k\colon S_k \to \mathbb{N}^k$ be constructed as follows. Given a set of k natural numbers $A = \{x_1 < x_2 < \ldots < x_k\}$ define $f_k(A) = (x_1, x_2, \ldots, x_k)$. By construction, f_k is 1-1. Thus $|S_k| \leq |\mathbb{N}^k| = |\mathbb{N}|$. By the theorem from class, $|S| \leq |\mathbb{N}|$ also. It's obvious that $|S| \geq |S_1| = |\mathbb{N}$. By Shroeder-Berenstein Theorem this implies that $|S| = |\mathbb{N}|$.

(2) Let S be an infinite set such that $|S| > |\mathbb{N}|$. Let $T \subset S$ be countable.
  (a) Prove that $S \backslash T$ is infinite.
  (b) Prove that $|S| = |S \backslash T|$.
  Hint: Construct $T' \subset S \backslash T$ such that T' is countable and use that $|T \cup T'| = |T|$ to construct an 1-1 and onto map from S to $S \backslash T$.
  (c) Find the cardinality of the set of transcendental numbers.

Solution

  (a) Suppose $A = S \backslash T$ is finite. Then $S = A \cup T$. Since $|A| \leq |\mathbb{N}|$ and $|T| \leq |\mathbb{N}|$ this implies that $|S| \leq |\mathbb{N}|$. This is a contradiction as we are given that $|S| > |\mathbb{N}|$.
  (b) Since $S \backslash T$ is infinite by part a), we can construct a countable subset $T' \subset S \backslash T$. Let $A = S \backslash (T \cup T')$.
  Note that $T \cap T'$ is countable since both T and T' are countable. Thus, $|T'| = |\mathbb{N}| = |T \cap T'|$. Therefore we can construct a 1-1 and onto map $f\colon T \cup T' \to T'$.
  Finally, define $F\colon S = T \cup T' \cup A \to S \backslash T = T' \cup A$ by the formula

$$F(s) = \begin{cases} f(s) \text{ if } x \in T \cup T' \\ s \text{ if } s \in A \end{cases}$$

  By construction, F is 1-1 and onto.
  (c) Let $S = \mathbb{R}$ and T be the set of all algebraic numbers. Then T is countable and $|S| = |\mathbb{R}| > |\mathbb{N}|$. The set of transcendental numbers is $S \backslash T$. Applying b) we conclude that $|S \backslash T| = |S| = |\mathbb{R}|$.

(3) Let S be the set of sequences $q_1, q_2, q_3, \ldots$ where q_i is real for every i and such that for every sequence there exists $n \in \mathbb{N}$ such that $q_i = 0$ for all $i \geq n$.

Find the cardinality of S.

Solution

Let S_n be the set of sequences of the form $q_1, \ldots, q_n, 0, 0, \ldots$. Then $S = \cup_{n=1}^{\infty} S_n$. Let $f_n\colon S_n \to \mathbb{R}^n$ be given by $f(q_1, \ldots, q_n, 0, 0, \ldots) = (q_1, \ldots, q_n)$. Clearly, f_n is 1-1 and hence, $|S_n| \leq |\mathbb{R}|$ for any n. It's also obvious that $|S_n| \geq |\mathbb{R}|$ for any n and therefore $|S_n| = |\mathbb{R}|$ for every n. By a theorem from class this implies that $|S| = |\mathbb{R}|$.

(4) Let $P(x)$ be a cubic polynomial with rational coefficients. Suppose it has a complex root of the form $a + bi$ where both a and b are rational.

Prove that $P(x)$ has a rational root.

Solution

First, we can assume that $b \neq 0$ since there is nothing to prove otherwise.

Let $Q(x) = (x-a-bi)(x-a+bi) = (x-a)^2+b^2 = x^2-2xa+a^2+b^2$ so $Q(x)$ has rational coefficients. Divide P by Q with a remainder. Then $P(x) = A(x)Q(x) + R(x)$ where both $A(x)$ and $R(x)$ have rational coefficients, $A(x)$ has degree 1 and $R(x)$ has degree at most 1.

Suppose $R(x)$ has degree exactly 1. That is $R(x) = a_1x + a_0$ where a_i are rational and $a_1 \neq 0$. Plugging in $x = a + bi$ into $P(x) = A(x)Q(x)+R(x)$ we get $0 = 0+R(a+bi)$ or $0 = a_1(a+bi)-a_0, a + bi = \frac{a_0}{a_1}$. This is a contradiction since $\frac{a_0}{a_1}$ is real and $a + bi$ is not.

Therefore, $R(x)$ has degree 0, i.e $R(x) = a_0$. Again plugging in $x = a + bi$ into $P(x) = A(x)Q(x) + R(x)$ we get $0 = 0 + a_0$ which means that $R(x) = 0$. Therefore $P(x) = A(x)Q(x)$.

Lastly, $A(x)$ is a degree 1 polynomial with rational coefficients. It obviously has a rational root which is also a root of $P(x)$.