Solutions to Practice Final 2

1. Using induction prove that

$$1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Solution

First we verify the base of induction. When $n = 0$ LHS$= 1^2 = 1$ and RHS$= \frac{1 \cdot 1 \cdot 3}{3} = 1$.
Induction step. Assume the formula is true for $n \geq 0$ and we need to verify it for $n+1$. Then we have

$$1^2 + 3^2 + \ldots + (2n+1)^2 + (2n+3)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} + (2n+3)^2 =$$
$$\frac{(n+1)(2n+1)(2n+3) + 3(2n+3)^2}{3} = \frac{(2n+3)(2n^2+3n+1+3(2n+3))}{3}$$
$$= \frac{(2n+3)(2n^2+9n+10)}{3} = \frac{(2n+3)(2n+5)(n+2)}{3}$$

This completes the induction step and proves the formula for all $n \geq 0$.

2. Let a, b, c be natural numbers.

   (a) Show that the equation $ax + by = c$ has a solution if and only if $(a, b)|c$.

   (b) Find all integer solutions of $6x + 15y = 9$.

Solution

(a) Suppose $ax + by = c$ for some integer x and y. If $d|a$ and $d|b$ then obviously, $d|ax + by = c$. In particular, if $(a, b)|c$.
Conversely, suppose $(a, b)|c$ so that $c = d \cdot (a, b)$. Then $ax + by = (a, b)$ has an integer solution by a result from class. Multiplying both sides by d we get $a(xd) + b(yd) = (a, b) \cdot d = c$.

(b) First, divide both sides by 3. we get $2x + 5y = 3$. We have $(2, 5) = 1$ and we can find integer solution of $2x + 5y = 1$ using either Euclidean algorithm or just by trying a few small numbers we get
$2 \cdot (-2) + 5 \cdot 1 = 1$. Multiplying by 3 we get $2 \cdot (-6) + 5 \cdot (3) = 3$ so $x_0 = -6, y_0 = 3$ is a solution of $2x + 5y = 3$.
It's easy to see that $x = -6 - 5k, y = 3 + 2k$ is a solution of $2x + 5y = 3$ for any k. We claim that any integer solution of $2x + 5y = 3$ has this form.
Suppose $2x + 5y = 3$. we also have $2 \cdot (-6) + 5 \cdot (3) = 3$. Subtracting these equations we get $2(-6 - x) + 5(3 - y) = 0$ or $2(-6 - x) = 5(y - 3)$. This implies that $2|(y - 3)$ so that $y - 3 = 2k$ or $y = 3 + 2k$. This gives $2(-6 - x) = 5(y - 3) = 6k, -6 - x = 3k, x = -6 - 3k$.
Thus the general solution is $x = -6 - 5k, y = 3 + 2k$ where k is any integer.

3. Find the last digit of the sum

$$2(1+3+3^2+3^3+\ldots+3^{309})$$

Solution

First, we compute

$$2(1+3+3^2+3^3+\ldots+3^{309}) = 2\cdot\frac{3^{310}-1}{3-1} = 3^{310}-1.$$

We have $\phi(10)=\phi(2\cdot 5)=1\cdot 4=4$. By Euler's theorem this implies that $3^4\equiv 1(\mod 10)$. Of course, this can also be seen directly as $3^4=81$.

Therefore $3^{4k}\equiv 1(\mod 10)$. We have $310=308+2$ and $4|308$. Therefore $3^{310}\equiv 3^2(\mod 10)$. This means that the last digit of 3^{310} is 9 and hence the last digit of $3^{310}-1$ is 8.

4. Let S be infinite and $A\subset S$ be finite. Prove that $|S|=|S\backslash A|$.

Solution

Let $A=\{s_1,\ldots,s_n\}$. Since S is infinite the set $S\backslash A$ is non empty. Pick any $s_{n+1}\in S\backslash A=S\backslash\{s_1,\ldots,s_n\}$. Next, since $S\backslash\{s_1,\ldots,s_{n+1}\}\neq\emptyset$ we can choose $s_{n+2}\in S\backslash\{s_1,\ldots,s_{n+1}\}$. Proceeding by induction we con construct $s_{m+1}\in S\backslash\{s_1,\ldots,s_m\}$ for any $m\geq n$.

Now define $f\colon S\to S\backslash A$ by the formula $f(s_i)=s_{i+n}$ for any i and $f(x)=x$ if $x\in S\backslash\{s_1,s_2,\ldots\}$. By construction, f is 1-1 and onto.

5. Let $S=[0,1]$ and $T=[0,2)$. Let $f\colon S\to T$ be given by $f(x)=x$ and $g\colon T\to S$ be given by $g(x)=x/2$.

   (a) Find S_S, S_T, S_∞;

   (b) give an explicit formula for a 1-1 and onto map $h\colon S\to T$ coming from f and g using the proof of the Cantor-Berenstein theorem.

Solution

(a) Note that $1\notin g(T)$ and therefore $1\in S_S$. Next, we see that $1/2\in S_S$ also. Indeed, $1/2=g(1)$ and $1=f(1)$. So 1 is the last ancestor of 1/2 and hence $1/2\in S_S$. proceeding by induction we see that $\frac{1}{2^n}\in S_S$ for any $n\geq 0$.

Next observe that $(1/2,1)\subset S_T$. Indeed, if $1/2<x<1$ then $x=g(2x)$ and $1<2x<2$ so that $2x\notin f(S)$.

Proceeding by induction we claim that $(\frac{1}{2^{n+1}}, \frac{1}{2^n}) \in S_T$ for any $n \geq 0$. We just verified the base of induction.

Induction step. Suppose we know the statement of $n \geq 0$ and we need to prove it for $n+1$. Let $\frac{1}{2^{n+2}} < x < \frac{1}{2^{n+1}}$ then $x = g(2x)$ and $\frac{1}{2^{n+1}} < 2x < \frac{1}{2^n}$. Also, $2x = f(2x)$. By induction assumption, $2x \in S_T$ and the last ancestor of x is the last ancestor of $2x$ so $x \in S_T$ also.

This concludes the induction step.

It's obvious that $0 \in S_\infty$. Therefore $S_\infty = \{0\}, S_S = \{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\}$ and $S_T = \{x \in [0,1]$ such that $x \neq 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$.

(b) By the proof of the Cantor-Berenstein Theorem the following map $h\colon S \to T$ is 1-1 and onto.

$$h(x) = \begin{cases} x \text{ if } x = 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \\ 2x \text{ if} x \neq 0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \end{cases}$$

6. Let $n = 2p$ where p is an odd prime. Find the remainder when $\phi(n)!$ is divided by n. Here $\phi(n)$ is the Euler function of n.

Solution

We have $\phi(n) = \phi(2p) = (2-1)(p-1) = p-1$. By Wilson's theorem $\phi(n)! = (p-1)! \equiv -1(\mod p) \equiv p-1(\mod p)$. This measn that $p|(p-1)! - (p-1)$. Since p is odd $p-1$ is even and therefore $2|(p-1)! - (p-1)$ also. Since $(2,p) = 1$ this implies that $2p|(p-1)! - (p-1)$ or, equivalently $(p-1)! \equiv p-1(\mod 2p)$.

Answer: $p-1$.

7. Prove that $q_1\sqrt{3} + q_2\sqrt{5} \neq q_1'\sqrt{3} + q_2'\sqrt{5}$ for any rational q_1, q_2, q_1', q_2' unless $q_1 = q_1', q_2 = q_2'$.

Solution

Suppose $q_1\sqrt{3} + q_2\sqrt{5} = q_1'\sqrt{3} + q_2'\sqrt{5}$. Then $(q_1 - q_1')\sqrt{3} + (q_2 - q_2')\sqrt{5} = 0$. Let $a = q_1 - q_1', b = q_2 - q_2'$ are rational and $a\sqrt{3} + b\sqrt{2} = 0$. We want to show that $a = b = 0$. If $a \neq 0$ this gives $\sqrt{\frac{3}{2}} = -\frac{b}{a}$ which is rational. This is a contradiction since $\sqrt{\frac{3}{2}}$ is irrational. Hence $a = 0$. Since $a\sqrt{3} + b\sqrt{2} = 0$ this implies $b\sqrt{2} = 0, b = 0$.

8. Let a be a root of $x^5 - 6x^3 + 2x^2 + 5x - 1 = 0$. Construct a polynomial with integer coefficients which has a^2 as a root.

Hint: separate even and odd powers.

Solution

We can rewrite the equation as $x^5 - 6x^3 + 5x = 1 - 2x^2, x(x^4 - 6x^2 + 5) = 1 - 2x^2$. Squaring both sides we get $x^2(x^4 - 6x^2 + 5)^2 = (1 - 2x^2)^2$. Clearly, $y = x^2$ satisfies $y(y^2 - 6y + 5)^2 = (1 - 2y)^2$.

9. Find all complex roots of $x^6 + 7x^3 - 8 = 0$.

   Reminder: Real numbers are also complex numbers.

Solution

Let $z = x^3$. Then z satisfies $z^2 + 7z - 8 = 0$ Solving this quadratic equation we get $z = 1, z = -8$. Thus we need to solve $x^3 = 1$ and $x^3 = -8$. Solving $x^3 = 1$ gives $x = 1, x = \cos(2\pi/3) + i\sin(2\pi/3) = \frac{-1+i\sqrt{3}}{2}, x = \cos(4\pi/3) + i\sin(4\pi/3) = \frac{-1-i\sqrt{3}}{2}$

Next we write -8 as $2^3(\cos\pi + i\sin\pi)$. Thus solving $x^3 = -8$ we get $x = 2(\cos(\pi/3) + i\sin(\pi/3)) = 1 + i\sqrt{3}, x = 2(\cos(\pi/3 + 2\pi/3) + i\sin(\pi/3 + 2\pi/3)) = 2(\cos\pi + i\sin\pi) = -2, x = 2(\cos(\pi/3 + 4\pi/3) + i\sin(\pi/3 + 4\pi/3)) = 2(\cos(5\pi/3) + i\sin(5\pi/3)) = 1 - i\sqrt{3}$

10. Represent $\sin(5\theta)$ as a polynomial in $\sin(\theta)$.

Solution

We have $\cos(5\theta) + i\sin(5\theta) = (\cos\theta + i\sin\theta)^5 = (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$ We compute separately $(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)$ and $(\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta) = (\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)(\cos\theta + i\sin\theta) = (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta + i(\cos^2\theta - \sin^2\theta)\sin\theta + 2i\sin\theta\cos^2\theta = \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta)$.

Combining these together we get $\cos(5\theta) + i\sin(5\theta) = (\cos\theta + i\sin\theta)^5 = (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3 = (\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)(\cos^3\theta - 3\sin^2\theta\cos\theta + i(3\sin\theta\cos^2\theta - \sin^3\theta)) = (\cos^2\theta - \sin^2\theta)(\cos^3\theta - 3\sin^2\theta\cos\theta) - 2\sin\theta\cos\theta(3\sin\theta\cos^2\theta - \sin^3\theta) + i(\cos^2\theta - \sin^2\theta)(3\sin\theta\cos^2\theta - \sin^3\theta) + 2i\sin\theta\cos\theta(\cos^3\theta - 3\sin^2\theta\cos\theta)$.

Therefore, $\sin(5\theta) = (\cos^2\theta - \sin^2\theta)(3\sin\theta\cos^2\theta - \sin^3\theta) + 2\sin\theta\cos\theta(\cos^3\theta - 3\sin^2\theta\cos\theta) = (1 - 2\sin^2\theta)(3\sin\theta(1 - \sin^2\theta) - \sin^3\theta) + 2\sin\theta\cos^4\theta - 6\sin^3\theta\cos^2\theta = (1 - 2\sin^2\theta)(3\sin\theta(1 - \sin^2\theta) - \sin^3\theta) + 2\sin\theta(1 - \sin^2\theta)^2 - 6\sin^3\theta(1 - \sin^2\theta)$.

11. Is $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ constructible? Justify your answer.

Solution

$\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is not constructible. We argue by contradiction. Assume $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is constructible. Since $\sqrt{5}$ and $\sqrt{7}$ are constructible this implies that $\sqrt[6]{5}$ is constructible and hence $(\sqrt[6]{5})^2 = \sqrt[3]{5}$ is also constructible. $\sqrt[3]{5}$ is a root of $x^3 - 5 = 0$ which is a cubic equation with integer coefficients. By a theorem from class if it has a constructible root it must have a rational root as well. Let $\frac{m}{n}$ be a rational root where $(m, n) = 1$. Then $m|5$ and $n|1$ which means that $\frac{m}{n} = \pm 1, \pm 5$. Plugging these numbers into $x^3 - 5 = 0$ we see that none of them are roots.

This is a contradiction and therefore $\frac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ is not constructible.

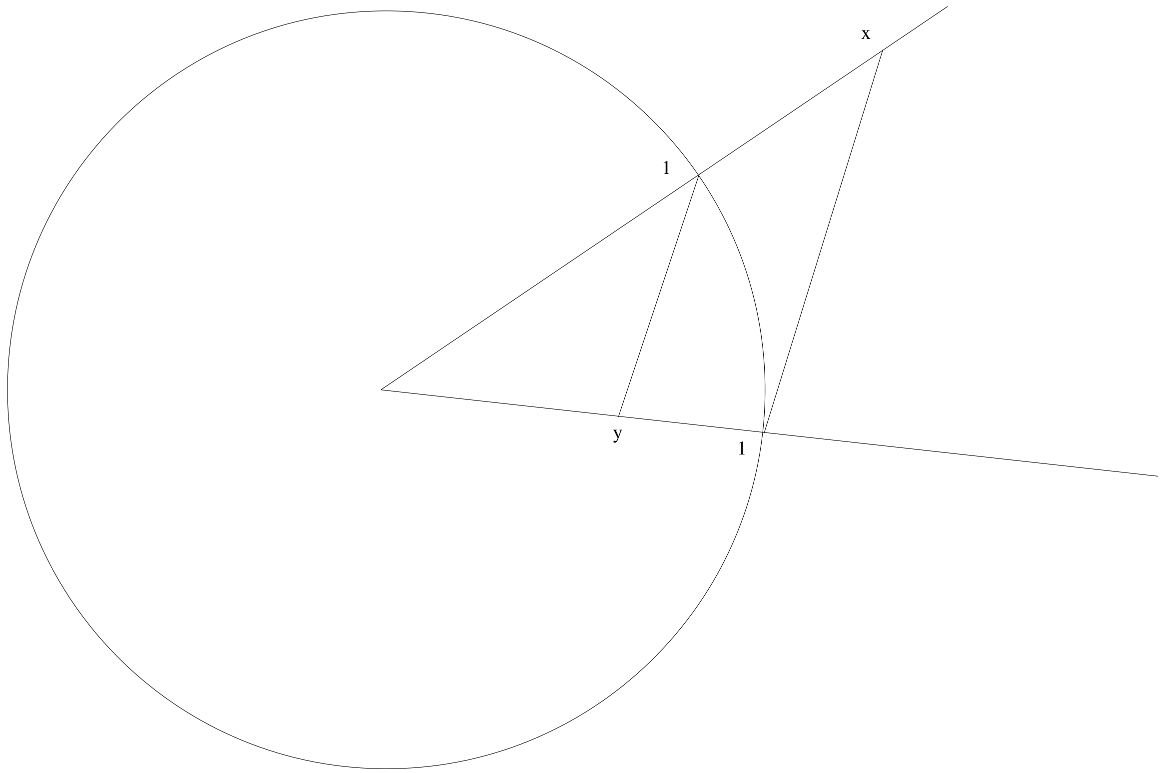
12. For each of the following answer "true" or "false". Justify your answer.

    a) If $\frac{x}{y}$ is constructible then both x and y are constructible.
    b) If x is constructible then $\frac{1}{x}$ is constructible.
    c) There is an angle θ such that $\cos\theta$ is constructible but $\sin\theta$ is not constructible.
    d) $\sqrt[3]{\frac{10}{27}}$ is constructible.

Solution

a) **False.** For example, take $x = y = \pi$. Then X and y are not constructible but $x/y = 1$ is constructible.

b) **True.** See figure below. Draw segments of lengths 1 and x on one side of an angle and a segment of length 1 on the other side. Connect x and 1 on opposite sides by a line a draw a parallel line through 1 on the same side as x. It intersect the second side of the angle at distance y. Then from similar triangles we get $\frac{x}{1} = \frac{1}{y}$ or $y = \frac{1}{x}$



c) **False.** If $\cos\theta$ is constructible then so is $1-\cos^2\theta$. Hence $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ is also constructible since a square root of a constructible number is constructible.

d) **False.** We argue by contradiction. Suppose $x = \sqrt[3]{\frac{10}{27}}$ is constructible. It satisfies the equation $27x^3 - 10 = (3x)^3 - 10 = 0$. If x is constructible then so is $y = 3x$ which satisfies the equation $y^3 - 10 = 0$. This is a cubic equation with integer coefficients. If it has a constructible root it must also have a rational one. We can write that rational root as $\frac{a}{b}$ where $(a,b) = 1$. Then $a|10$ and $b|1$ which means that $y = \frac{a}{b} = \pm 1 \pm 2 \pm 5$ or ± 10. By plugging these numbers into $y^3 - 10 = 0$ we see that none of them are roots. This is a contradiction and therefore $\sqrt[3]{\frac{10}{27}}$ is not constructible.

13. Prove that the equation

$$(1+x^{19})^3 + (1+x^{19})^2 - 3 = 0$$

has no constructible solutions.

Solution

Suppose x is a constructible root. Then $y = x^{19} + 1$ is also constructible and it satisfies $y^3 + y^2 - 1 = 0$. This is a cubic equation with integer coefficients. If it has a constructible root it must also have a rational one. We can write that rational root as $\frac{a}{b}$ where $(a, b) = 1$. Then $a|1$ and $b|1$ which means that $y = \frac{a}{b} = \pm 1$. But neither $y = 1$ nor $y = -1$ solve $y^3 + y^2 - 1 = 0$. This is a contradiction which means that $(1 + x^{19})^3 + (1 + x^{19})^2 - 3 = 0$ has no constructible solutions.