Practice Final 4

1. Use induction to prove that

$$1 + 2q + 3q^2 + \ldots + nq^{n-1} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)^2}$$

for any real $q \neq 1$ and any natural n.

2. (a) Find $43!$ (mod 45).
   (b) Find the last digit of 3^{2014}.

3. Let p_1, p_2 be distinct prime numbers.
   Prove the formula

$$\phi(p_1^{k_1} p_2^{k_2}) = (p_1^{k_1} - p_1^{k_1-1})(p_2^{k_2} - p_2^{k_2-1})$$

where ϕ is Euler's function.
You are not allowed to use any theorems about Euler's function in the proof.

4. Prove that for any complex numbers z_1, z_2 the following equalities hold
   (a) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
   (b) $|z_1 z_2| = |z_1| \cdot |z_2|$

5. Find all complex solutions of the following equation

$$z^6 + 7z^3 - 8 = 0$$

6. Let S be the set of all functions $g : \mathbb{Z} \longrightarrow \{1, 2\}$.
   Is S countable? Justify your answer.

7. Construct an explicit 1-1 and onto map from $S = (0, 1)$ to $T = [0, 1]$.

8. **Mark True or False**. You do NOT need to justify your answer.
   (a) Product of two constructible numbers is constructible.
   (b) If x, y are constructible then x^y is constructible.
   (c) The set of non constructible numbers is a number field.
   (d) If x is not constructible then x is irrational
   (e) If x is not constructible then x is transcendental.

9. Which of the following are constructible. **Justify your answer**.

   (a) $\tan(\pi/30)$

   (b) $\dfrac{\sqrt[6]{8}}{\sqrt[4]{3/5+\sqrt{1+\sqrt{1.3}}}}$

10. A message was encoded using the RSA encryption. The encoded message is $R = 19$. Decode the original message M, if $N = 21$, $E = 5$.