1. Setting $k = 1$, we get $1 = \frac{1-(1+1)q+(1)q^2}{(1-q)^2}$

Which we see is true as $(1-q)^2 = 1 - 2q + q^2$ and $q \neq 1$
Assume the statement is true for $k = n$
i.e. $1 + 2q + 3q^2 + ... + nq^{n-1} = \frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$
Then we have,
$1 + 2q + 3q^2 + ... + nq^{n-1} + (n+1)q^n = \frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2} + (n+1)q^n$

$$= \frac{1-(n+1)q^n + nq^{n+1} + (1-2q+q^2)(n+1)q^n}{(1-q)^2} = \frac{1-(n+2)q^{n+1} + (n+1)q^{n+2}}{(1-q)^2}$$

QED

2.a. $45 = 9 * 5$. Since 9 and 5 are both less than 43, both occur as separate factors in 43!, or $43! = 43 * 42 * 41 * .... * 9 * ... * 5 * .. * 1$

Therefore $43! \equiv 0 \pmod{45}$

2.b. $3^2 \equiv 9 \equiv -1$ mod 10.
Thus, $3^{2014} \equiv (-1)^{1007} \equiv -1 \equiv 9$ mod 10
i.e. 9 is the last digit of the number

3. Instead of $p_1^{k_1}p_2^{k_2}$ we shall refer it to as $n = p^a q^b$. Now any divisor of n will look like $p^r q^s$ where $r \le a, s \le b$. Thus, if any number had gcd not 1 with n, then the gcd which is a divisor of n, should divide the chosen number. That is $p^r q^s$ divides the number where not both r and s are 0.i.e either p divides the number or q divides the number,(or both in which case it is divisible by pq as they are coprime).
The number of multiples of p less than or equal to n is $\frac{n}{p} = p^{a-1}q^b$, given by $\{p*1, p*2, ...p*\frac{n}{p}\}$
Similarly. The number of multiples of q less than or equal to n is $\frac{n}{q} = p^a q^{b-1}$, given by $\{q*1, q*2, ...q*\frac{n}{q}\}$
Similarly. The number of multiples of pq less than or equal to n is $\frac{n}{pq} = p^{a-1}q^{b-1}$, given by $\{pq*1, pq*2, ...pq*\frac{n}{qp}\}$
So the number of numbers which are divisible by p or q will be $p^{a-1}q^b + p^a q^{b-1}$, but we have counted the multiples of pq twice, so we subtract them once. i.e. $p^{a-1}q^b + p^a q^{b-1} - p^{a-1}q^{b-1}$
So, $\phi(n) = n - (p^{a-1}q^b + p^a q^{b-1} - p^{a-1}q^{b-1})$
$= p^a q^b - p^{a-1}q^b - p^a q^{b-1} + p^{a-1}q^{b-1}$
$= (p^a - p^{a-1})(q^b - q^{b-1})$

4. Let $z_1 = a + ib$ and $z_2 = c + id$, where a, b, c, d are reals. Then,
a. $(a-ib)(c-id) = (ac-bd) - i(ad+bc) = \overline{ac - bd + i(ab+bc)} = \overline{(a+ib)(c+id)}$

b. $|z_1z_2|^2 = (z_1z_2)\overline{z_1z_2} = z_1z_2\bar{z}_1\bar{z}_2 = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2$.
Since $|z|$ is always a positive real, this implies $|z_1z_2| = |z_1||z_2|$

5. Let $z^3 = t$, then the equation turns to $0 = t^2 + 7t - 8 = (t-1)(t+8)$.
Case 1 $t - 1 = 0$, then $z^3 = t = 1$. Obviously we see that $z = 1$ is a solution.
Thus 1 is a root of $z^3 - 1$, thus,$z^3 - 1 = (z-1)(z^2 + z + 1)$.
Solving the quadratic we get that $z = \frac{-1 \pm i\sqrt{3}}{2}$ are also roots.
Case 2 $t + 8 = 0$, setting $-2s = z$, equation becomes $-8s^3 + 8 = 0$ or $s^3 - 1 = 0$
This from case one we know imples $\frac{z}{-2} = s = 1, \frac{-1 \pm i\sqrt{3}}{2}$
Thus all complex values for z are $1, \frac{-1 \pm i\sqrt{3}}{2}, -2, 1 \pm i\sqrt{3}$

6. We can give a bijection from the power set of Z to the set of functions $\{g : \mathbb{Z} \longrightarrow \{1, 2\}\}$ as follows. For any $S \subset Z$ We define $f_S(t) = 1$ if $t \in S$ and $f_S(t) = 2$ otherwise.
To see that this is a bijection, we define another function that takes functions to subsets of Z, given by taking a function f to the set $f^{-1}(1) = \{t \in Z | f(t) = 1\}$.
It is easy to see that these processes are inverses of each other, proving a bijection.

7. $f : S \longrightarrow T$, we define by
$f(x) = \frac{x}{x-1}$ if $nx = 1$ for some $n \in N$,
$f(\frac{2}{3}) = 0$
$f(x) = \frac{x}{2-2x}$ if $(2n+1)x = 2$ for some $n \in N$ and $n \neq 1$
$f(x) = x$, otherwise
What we did was shifted the sequence $\frac{1}{n}$ from $n = 2$ onward to right by 1, and added one at the start
shifted the sequence $\frac{2}{2n+1}$ to the right and fit 0 as the first term.

8. a True (constructible numbers form a field)
b False ($2^{\frac{1}{3}}$ is not constructible)
c. False (it contains transcedental numbers)
d True (all rationals are constructible)
e. False ($2^{\frac{1}{3}}$ is not constructible)

9.a $tan(\pi/30)$ is constructible if the angle $\pi/30 = 6^o$ is constructible. Since 3^o is constructible(Textbook theorem), $6^o = 2 * 3^o$ is constructible.

9.b $\sqrt[6]{8} = \sqrt{2}$, therefore is constructible. $\sqrt{1.3}$ is constructible(1.3 is constructible). And $sqrt3/5$ is constructible, which implies, it square root $\sqrt[4]{3/5}$ is constructible. Thus, $\sqrt[4]{3/5} +$

$\sqrt{1.3}$ is constructible. And thus its inverse is constructible and thus $\sqrt{2}\frac{1}{\sqrt[4]{3/5}+\sqrt{1.3}}$ is constructible.

10. $N = 21 = 3 * 7$, $\phi(N) = (3-1)(7-1) = 12$
$ED \equiv 1 \bmod \phi(N)$
$5D \equiv 1 \bmod 12$, $5D \equiv 25 \bmod 12$.
i.e. $D \equiv 5 \bmod 12$ or $D = 5$
$R^D \equiv M \bmod N$
As $19 \equiv -2 \bmod 21$,
$19^5 \equiv (-2)^5 \equiv -32 \equiv 10 \bmod 21$
Thus, $M = 10$