(1) Prove that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$$

(2) Prove that

$$1+2q+3q^2+\ldots+nq^{n-1}=\frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$$

(3) The Fibonacci sequence is the sequence of numbers $F(0), F(1), \ldots$ defined by the following recurrence relations:
$F(0)=1, F(1)=1$, $F(n)=F(n-1)+F(n-2)$ for all $n>1$.
For example, the first few Fibonacci numbers are $1,1,2,3,5,8,13,\ldots$

Prove that

$$F(n)=\frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}-\Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big]$$

for all $n\geq 0$.

Hint: The computations can be simplified by using the fact that both of the numbers $x=\frac{1+\sqrt{5}}{2}$ and $x=\frac{1-\sqrt{5}}{2}$ satisfy the equation $1+x=x^2$.

(4) Prove that

$$\frac{5}{1\cdot 2\cdot 3}+\frac{6}{2\cdot 3\cdot 4}+\frac{7}{3\cdot 4\cdot 5}+\ldots+\frac{n+4}{n(n+1)(n+2)}=\frac{n(3n+7)}{2(n+1)(n+2)}$$

for any $n\geq 1$.

(5) Using the method from class find the formula for the sum

$$1^3+2^3+\ldots+n^3$$

Then prove the formula you've found by mathematical induction.

(6) Find a mistake in the following "proof".

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in n: Any two natural numbers $\leq n$ are equal.

We prove it by induction in n.

a) The statement is trivially true for $n=1$.

b) Suppose it's true for $n\geq 1$. Let a,b be two natural numbers $\leq n+1$. Then $a-1\leq n$ and $b-1\leq n$. Therefore, by the induction assumption

$$a-1=b-1$$

Adding 1 to both sides of the above equality we get that $a=b$. Thus the statement is true for $n+1$. By the principle of mathematical induction this means that it's true for all natural n. $\square$.