Solutions to selected problems from homework 1

(1) The Fibonacci sequence is the sequence of numbers $F(0), F(1), \ldots$ defined by the following recurrence relations:
$F(0) = 1, F(1) = 1,\ F(n) = F(n-1) + F(n-2)$ for all $n > 1$.
For example, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \ldots$

Prove that

$$F(n) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big]$$

for all $n \geq 0$.

Solution

We prove the formula by induction on n.

First let's check that the formula holds for $n = 0$ and $n = 1$.

For $n = 0$ we have $F(0) = 1$ and $\frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^1 - \Big(\frac{1-\sqrt{5}}{2}\Big)^1\Big] = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}-(1+\sqrt{5})}{2} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$.

Thus the formula holds for $n = 0$.

For $n = 1$ we have $F(1) = 1$ and $\frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^2 - \Big(\frac{1-\sqrt{5}}{2}\Big)^2\Big] = \frac{1}{\sqrt{5}}\frac{6+2\sqrt{5}-(6-2\sqrt{5})}{4} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1$.

Thus the formula also holds for $n = 1$.

Induction step. Suppose the formula holds for all $k = 0, 1, \ldots n$ for some $n \geq 1$. We need to show that

$$F(n+1) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+2} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+2}\Big]$$

Using that $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ satisfy the equation $1 + x = x^2$ we compute

By the induction assumption

$$F(n+1) = F(n-1) + F(n) = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n}\Big] + \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big]$$

$$= \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n}(1 + \frac{1+\sqrt{5}}{2}) - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n}(1 + \frac{1-\sqrt{5}}{2})\Big]$$

$$= \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n}(\frac{1+\sqrt{5}}{2})^2 - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n}(\frac{1-\sqrt{5}}{2})^2\Big]$$

$$= \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+2} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+2}\Big]$$

(2) Using the method from class find the formula for the sum

$$1^3 + 2^3 + \ldots + n^3$$

Then prove the formula you've found by mathematical induction.

Solution

We've proved in class that $1+2+\ldots+n=\frac{n(n+1)}{2}$ and $1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$.

Let's find a_n such that $a_1+a_2+\ldots+a_n=n^4$. We have $n^4=(a_1+\ldots+a_{n-1})+a_n=(n-1)^4+a_n$ and hence $a_n=n^4-(n-1)^4=n^4-(n^4-4n^3+6n^2-4n+1)=4n^3-6n^2+4n-1$.

Thus

$$n^4=(4\cdot1^3-6\cdot1^2+4\cdot1-1)+(4\cdot2^3-6\cdot2^2+4\cdot2-1)+\ldots+(4\cdot n^3-6\cdot n^2+4\cdot n-1)=$$

$$=4(1^3+2^3+\ldots+n^3)-6(1^2+2^2+\ldots n^2)+4(1+2+\ldots+n)-n=$$

$$=4(1^3+2^3+\ldots+n^3)-6\cdot\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}-n$$

Therefore,

$$4(1^3+2^3+\ldots+n^3)=n^4+n(n+1)(2n+1)-2n(n+1)+n$$

$$1^3+2^3+\ldots+n^3=\frac{n^4+n(n+1)(2n+1)-2n(n+1)+n}{4}=\Big(\frac{n(n+1)}{2}\Big)^2$$

Now that we have found the formula we can also prove it by induction.

First check that it holds for $n=1$: $1^3=1=(\frac{1\cdot2}{2})^2$. This verifies the base of induction.

Induction step: Suppose $1^3+2^3+\ldots+n^3=\big(\frac{n(n+1)}{2}\big)^2$ for some $n\geq1$.

We need to show that $1^3+2^3+\ldots+n^3+(n+1)^3=\big(\frac{(n+1)(n+2)}{2}\big)^2$.

We have $1^3+2^3+\ldots+n^3+(n+1)^3=\big(\frac{n(n+1)}{2}\big)^2+(n+1)^3=\big(\frac{(n+1)(n+2)}{2}\big)^2$. $\square$.

(3) Find a mistake in the following "proof".

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in n: Any two natural numbers $\leq n$ are equal.

We prove it by induction in n.

a) The statement is trivially true for $n=1$.

b) Suppose it's true for $n\geq1$. Let a,b be two natural numbers $\leq n+1$. Then $a-1\leq n$ and $b-1\leq n$. Therefore, by the induction assumption

$$a-1=b-1$$

Adding 1 to both sides of the above equality we get that $a=b$. Thus the statement is true for $n+1$. By the principle of mathematical induction this means that it's true for all natural n. $\square$.

Solution

The mistake is in step b) in the implication that since $a-1 \leq n, b-1 \leq n$ they must be equal by the induction assumption. The induction assumption is only valid for natural numbers which are all ≥ 1. However $a-1, b-1$ need not natural. one or both of them can be 0 which is not a natural number and therefore the induction assumption need not be applicable to $a-1, b-1$. For example this happens if $n=1, a=1, b=2$. Then $a-1=0, b-1=1$.

(4) #14 from the book.

Solution

(a) We have that $F_n = 2^{2^n}+1$ for $n = 0, 1, \ldots$ and we need to show that $F_0 \cdot \ldots \cdot F_{n-1} + 2 = F_n$ for $n \geq 1$. We do this by induction in n.
First we check the formula for $n=1$. $F_0 = 2^{2^0}+1 = 2^1+1 = 3, F_1 = 2^{2^1}+1 = 5$ and $F_0 + 2 = 3+2 = 5 = F_1$. Thus the formula holds for $n=1$.
Induction step. Suppose we know that $F_0 \cdot \ldots \cdot F_{n-1} + 2 = F_n$ for some $n \geq 1$. We need to show that $F_0 \cdot \ldots \cdot F_n + 2 = F_{n+1}$. We have $F_{n+1} = 2^{2^{n+1}}+1 = 2^{2 \cdot 2^n}+1 = (2^{2^n})^2+1 = (F_n - 1)^2+1 = F_n^2 - 2F_n + 1 + 1 = F_n(F_n-2)+2$. By the induction assumption $F_n - 2 = F_0 \cdot \ldots \cdot F_{n-1}$ and therefore
$F_{n+1} = (F_n-2)F_n + 2 = F_0 \cdot \ldots \cdot F_{n-1} \cdot F_n + 2$.

(b) Let us first prove the following
Claim F_n, F_m have no common prime factors for $n \neq m$.
Let $m < n$. Suppose p is a common prime factor for both F_n, F_m. Then $p|F_n$ and $p|F_0 \ldots F_{n-1}$ because the latter product contains F_m as a factor. Thus we can write $F_n = ap, F_0 \ldots F_{n-1} = bp$ for some natural a, b. By part a) we know that $F_0 \ldots F_{n-1} + 2 = F_n$ which means that $bp + 2 = ap$. Hence $2 = ap - bp = p(a-b)$ and p divides 2. Therefore $p = 2$ since that's the only divisor of 2 bigger than 1. However p can not be 2 since all Fermat numbers are odd. This is a contradiction and hence F_n, F_m have no common prime divisors. This proves the Claim.
Next, for any $n \geq 1$ pick p_n to be some prime divisor of F_n. This gives us a sequence of prime numbers $p_1, p_2, p_3, \ldots$. By the Claim above they are all distinct which implies that the set of prime numbers is infinite.