(1) Using the Euclidean Algorithm prove that if $gcd(a, b) = 1$ and $a|c, b|c$ then $ab|c$.
As $gcd(a, b) = 1$ we know by euclidean Algorithm that there exists integers m, n such that $am + bn = 1$. Multiplying this equation by c we get, $acm + bcn = c$. Now Since, $a|c$, we have $ab|bc$ and since $b|c$ we have $ab|ac$. Thus ab divides both the terms on the left hand side. Thus $ab|rhs = c$.

(2) Using the Euclidean Algorithm find $gcd(291, 573)$ and integer x, y such that $291x + 573y = gcd(291, 573)$.
$573 = 291 * 1 + 282$
$291 = 282 * 1 + 9$
$282 = 9 * 31 + 3$
$9 = 3 * 3 + 0$
thus 3 is the gcd. And we have $573-291 = 282$ thus substituting in second gives $291-(573-291) = 2*291-573 = 9$ substituting the first 2 in third gives $573-291 = 282 = 31(2*291-573)+3$ i.e. $3 = 32*573-63*291$.

(3) (a) Find all *integer* solutions of the equation

$$25x + 10y = 200$$

divide the whole equation by 5 to get $5x+2y = 40$. Clearly $5*8+2*0 = 40$. Now suppose for some other x and y the equation is satisfies. Then we have $5x + 2y = 40 = 5 * 8 + 2 * 0$ i.e $5(x - 8) = -2y$
Since 2 and 5 are coprime we have $5|y$ and $2|x-8$. Let $x = 2a+8$ and $y = 5b$ substituting in the equation be get $a = -b$ thus all solutions are given by $x = 2a + 8$ and $y = -5a$

(b) Find all *natural* solutions of the equation

$$25x + 10y = 200$$

divide the whole equation by 5 to get $5x + 2y = 40$ since we are looking for naturaal solutions we want $x = 2a + 8 \geqslant 1$ i.e. $a \geqslant \frac{-7}{2}$ and $y = -5a \geqslant 1$ i.e. $a \leqslant \frac{-1}{5}$. As a in an integer it means the only values it can take is $-3, -2, -1$ each of which will give a natural solution. $a = -3$ gives $x = -6 + 8 = 2, y = -5 \cdot (-3) = 15$
$a = -2$ gives $x = -4 + 8 = 4, y = 10$
$a = -1$ gives $x = -2 + 8 = 6, y = 5$.
Answer: The natural solutions are $(2, 15), (4, 10), (6, 5)$.

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1a,d
a. $252 = 2^2 * 63 = 2^2 * 3^2 * 7$ and $198 = 2 * 99 = 2 * 3^2 * 11$,
thus gcd is $2 * 3^2 = 18$
d. $52 = 2^2 * 13$ and $135 = 3^3 * 5$ thus gcd is 1.

5b,d
b $\phi(26 = 2 * 13) = (2 - 1)(13 - 1) = 12$
d $\phi(36 = 2^2 * 3^2) = 2 * 3 * (2 - 1) * (3 - 1) = 12$

8. We want to solve $24x \equiv 2 \mod 59$ i.e.(dividing by 2) $12x \equiv 1 \mod 59$ i.e. $12x \equiv 1 + 59 = 60 \mod 59$ i.e.(dividing by 12) $x \equiv 5 \mod 59$. So the smallest x will be 5.

11. Let $(a, b) \neq 1$ then there exists a prime $p|(a, b)$ i.e. $p|a$ and $p|b$ $\implies p|a^n$ and $p|b^n \implies p|(a^n, b^n) = 1$. Which is a contradiction. Thus our assumption $(a, b) \neq 1$ was wrong.

13a. let $(a, b) = d$ then $d|a$ and $d|b$ therefore $d|am + bn = 1 \implies d = 1$
13b. $(7a+3)5+(5a+2)(-7) = 1$ therefore by part a, their gcd is one,i.e. they are relatively prime.

19. First off we note that $(m, n) = 1$ i.e. $\exists A, B$ such that $mA + nB = 1$.
Now we want $x \equiv a \mod m$ or $x = mk + a$ and $x \equiv b \mod n$ or $x = nl + b$. i.e we want integers k and l such that
$mk + a = x = nl + b$ or $mk - nl = b - a$,
To get such k and l, we just multiply our $mA + nB = 1$ by $b - a$,
which will give $k = (b - a)A$ and $l = B(a - b)$.
Thus now we set $x = mk + a$ which will be equal to $nl + b$ by our definition.
This x will thus satisfy the congruences.