(1) Let a, b be odd integers.

Prove that $\sqrt{a^2+b^2}$ is irrational.

Hint: Look at divisibility by the powers of 2.

Solution

Suppose $\sqrt{a^2+b^2}$ is rational. By a theorem prove in class. $\sqrt{n}$ is rational if and only if n is a complete square. Thus $a^2+b^2=c^2$ for some natural c. We claim that it is impossible. we check divisibility by 4. Since a, b are odd a^2, b^2 are also odd and hence $c^2=a^2+b^2$ is even. Therefor c is even. Hence $c=2c_1$, $c^2=4c_1^2$ and 4 divides c^2.

On the other hand if for any odd number $2n+1$ we have $(2n+1)^2 = 4n^2+4n+1 \equiv 1 \pmod 4$. Thus $a^2 \equiv 1 \pmod 4$, $b^2 \equiv 1 \pmod 4$ and therefore $a^2+b^2 \equiv 1+1 \equiv 2 \pmod 4$. This means that $c^2=a^2+b^2$ is not divisible by 4. This is a contradiction and hence $\sqrt{a^2+b^2}$ is irrational. $\square$.

(2) Prove that for any real numbers $a < b$ there exists an irrational number c such that $a < c < b$.

Hint: Look at the numbers of the form $q\sqrt{2}$ where q is rational.

Solution

First, by making the interval (a, b) smaller if necessary we can assume that $0 \notin (a, b)$.

We know that $\sqrt{2}$ is irrational and hence $\frac{m}{n}\sqrt{2}$ is irrational for any integer m, n with $m \neq 0, n \neq 0$.

Take $n > \frac{\sqrt{2}}{b-a}$. Then $\frac{\sqrt{2}}{n} < \frac{b-a}{2}$. We claim that for some integer $m \neq 0$ we have that $\frac{m}{n}\sqrt{2} \in (a, b)$.

To see this, take the largest m such that $\frac{m}{n}\sqrt{2} \leq a$. Then $\frac{m+1}{n}\sqrt{2} > a$. But it can not be bigger than or equal to b because otherwise

$\frac{m}{n}\sqrt{2} \leq a < b \leq \frac{m+1}{n}\sqrt{2}$. Therefore, $b-a < \frac{m+1}{n}\sqrt{2} - \frac{m}{n}\sqrt{2} = \frac{\sqrt{2}}{n}$. This contradicts the choice of n. Therefore $a < \frac{m}{n}\sqrt{2} < b$. lastly, $m \neq 0$ since $0 \notin (a, b)$.

(3) Suppose $5+4i = (a+bi)(c+di)$ where a, b, c, d are integers. Prove that $|a+bi| = 1$ or $|c+di| = 1$.

Hint: use that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Solution

We have $|5+4i| = |a+bi| \cdot |c+di|$ or $\sqrt{5^2+4^2} = \sqrt{41} = \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}$. Taking squares of both sides we get

$$41 = (a^2+b^2) \cdot (c^2+d^2)$$

Since 41 is prime and both a^2+b^2 and c^2+d^2 are natural numbers, one of the factors in the above formula must be equal to 1. Therefore, $a^2+b^2=1$ or $c^2+d^2=1$ and hence $\sqrt{a^2+b^2}=1$ or $\sqrt{c^2+d^2}=1$.

(4) Let $P(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial with real coefficients.

Prove that if z_0 is a root of $P(z)=0$ then $\bar{z}_0$ is also a root of $P(z)=0$.

Solution

Recall that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$. for $z_1 = z_2 = z$ this gives $\overline{z^2} = \bar{z}^2$. By induction this easily yields $\overline{z^n} = \bar{z}^n$ for any natural n. Also, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Lastly, recall that $\bar{x} = x$ for any real x.

We have that $0 = P(z_0) = a_n z_0^n + \ldots + a_1 z_0 + a_0$. Taking bars of both sides we get

$0 = \bar{0} = \overline{a_n z_0^n + \ldots + a_1 z_0 + a_0} = \overline{a_n z_0^n} + \ldots + \overline{a_1 z_0} + \bar{a}_0 = \bar{a}_n \overline{z_0^n} + \ldots + \bar{a}_1 \bar{z}_0 + \bar{a}_0 = \bar{a}_n \bar{z}_0^n + \ldots + \bar{a}_1 \bar{z}_0 + \bar{a}_0 = a_n \bar{z}_0^n + \ldots + a_1 \bar{z}_0 + a_0$ where on the last step we used that all a_k are real and hence $\bar{a}_k = a_k$. Thus, $0 = a_n \bar{z}_0^n + \ldots + a_1 \bar{z}_0 + a_0$, i.e. $P(\bar{z}_0) = 0$.

(5) #11b on page 69: We need to show that $x = \sqrt[3]{4} + \sqrt{10}$ is irrational.

Solution

Suppose not and x is rational. Then $x - \sqrt{10} = \sqrt[3]{4}$. taking cubes of both sides we get $4 = (x - \sqrt{10})^3 = x^3 - 3x^2\sqrt{10} + 3x\sqrt{10}^2 - \sqrt{10}^3 = x^3 - 3x^2\sqrt{10} + 30x - 10\sqrt{10}$. Then $x^3 + 30x - 4 = (3x^2 + 10)\sqrt{10}$. Since $3x^2 + 10 > 0$ we can divide by $3x^2 + 10$ and get $\sqrt{10} = \frac{x^3+30x-4}{3x^2+10}$. Since x is rational $\frac{x^3+30x-4}{3x^2+10}$ is also rational and hence $\sqrt{10}$ is rational. This is a contradiction since 10 is not a complete square and hence $\sqrt{10}$ is irrational.

Thus, $x = \sqrt[3]{4} + \sqrt{10}$ is irrational. $\square$

(6) #13 on page 69: We need to show that for any natural n, k we have that $\sqrt[k]{n}$ is rational if and only if it's an integer.

In other words, $\sqrt[k]{n}$ is rational if and only if n is a complete k-th power.

Solution

Suppose $\sqrt[k]{n}$ is an integer. Then it's rational since integers are rational.

Conversely, suppose $x = \sqrt[k]{n}$ is rational. Then x satisfies $x^k = n, x^k - n = 0$. If x is rational it can be written as $\frac{p}{q}$ where $gcd(p,q) = 1$. Then, by the rational root theorem, $p| - n, q|1$.Thus, $q = \pm 1$ and hence $x = \frac{p}{\pm 1} = \pm p$ is an integer. $\square$.