(1) #3 on page 83:
   (a) find $\sqrt{-i}$
   (b) find $\sqrt{-15-8i}$.

**Solution**

(a) since $|-i| = 1$ we write $-i = 1 \cdot (\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$. Therefore, $\sqrt{-i} = \pm(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}) = \pm(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$ **Answer:** $\sqrt{-15-8i} = \pm(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$.

(b) Firs we compute $|-15-8i| = \sqrt{15^2+8^2} = \sqrt{289} = 17$. Hence we can write $-15-8i$ as $-15-8i = 17(-\frac{15}{17} - i\frac{8}{17}) = 17(\cos\theta + i\sin\theta)$ where $\cos\theta = -\frac{15}{17}$ and $\sin\theta = -\frac{8}{17}$. Note that this means that $\pi < \theta < 3\pi/2$.
Then $\sqrt{-15-8i} = \pm\sqrt{17}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$. Since $\pi < \theta < 3\pi/2$ we have that $\pi/2 < \theta < 3\pi/4$ and hence $\cos\frac{\theta}{2} < 0$, $\sin\frac{\theta}{2} > 0$.
Using the formula $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ we get $2\cos^2\frac{\theta}{2} - 1 = -\frac{15}{17}$, $\cos^2\frac{\theta}{2} = \frac{1}{17}$, $\cos\frac{\theta}{2} = -\frac{1}{\sqrt{17}}$. since $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$ this gives $\sin\frac{\theta}{2} = \frac{4}{\sqrt{17}}$.
Thus, $\sqrt{17}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) = \sqrt{17}(-\frac{1}{\sqrt{17}} + i\frac{4}{\sqrt{17}}) = -1+4i$.
**Answer:** $\sqrt{-15-8i} = \pm(-1+4i)$.

(2) #7 on page 83: Find all solutions of $iz^2 + 2z + i = 0$.

**Solution**

By the general formula for solving $az^2 + bz + c = 0$ we have $z = \frac{-1\pm\sqrt{1^2-i^2}}{i} = \frac{-1\pm\sqrt{2}}{i} = i \pm i\sqrt{2}$.
**Answer:** $i \pm i\sqrt{2}$

(3) #9 on page 83: Solve $z^6 + z^3 + 1 = 0$.

**Solution**

Let $x = z^3$. Then $x$ satisfies $x^2 + x + 1 = 0$ so $x = \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{3}i}{2}$. We have two possibilities

**1)** $x = \frac{-1+\sqrt{3}i}{2} = \cos(2\pi/3) + i\sin(2\pi/3)$. Solving $z^3 = x = \cos(2\pi/3) + i\sin(2\pi/3)$ we get $z = \cos(2\pi/9 + \frac{2\pi k}{3}) + i\sin(2\pi/9 + \frac{2\pi k}{3})$ where $k = 0, 1, 2$. This gives 3 solutions
when $k = 0$ we get $z_1 = \cos(2\pi/9) + i\sin(2\pi/9)$
when $k = 1$ we get $z_2 = \cos(2\pi/9 + \frac{2\pi}{3}) + i\sin(2\pi/9 + \frac{2\pi}{3}) = \cos(\frac{8\pi}{9}) + i\sin(\frac{8\pi}{9})$
when $k = 2$ we get $z_3 = \cos(2\pi/9 + \frac{4\pi}{3}) + i\sin(2\pi/9 + \frac{4\pi}{3}) = \cos(\frac{14\pi}{9}) + i\sin(\frac{14\pi}{9})$

**2)** $x = \frac{-1-\sqrt{3}i}{2} = \cos(4\pi/3) + i\sin(4\pi/3)$. Solving $z^3 = x = \cos(4\pi/3) + i\sin(4\pi/3)$ we get $z = \cos(4\pi/9 + \frac{2\pi k}{3}) + i\sin(4\pi/9 + \frac{2\pi k}{3})$ where $k = 0, 1, 2$. As before, this gives 3 solutions
when $k = 0$ we get $z_4 = \cos(4\pi/9) + i\sin(4\pi/9)$

when $k=1$ we get $z_5=\cos(4\pi/9+\frac{2\pi}{3})+i\sin(4\pi/9+\frac{2\pi}{3})=\cos(\frac{10\pi}{9})+i\sin(\frac{10\pi}{9})$
when $k=2$ we get $z_6=\cos(4\pi/9+\frac{4\pi}{3})+i\sin(4\pi/9+\frac{4\pi}{3})=\cos(\frac{16\pi}{9})+i\sin(\frac{16\pi}{9})$

(4) #14 on page 84: Show that every non-constant polynomial $p(z)=a_nz^n+\ldots+a_1z+a_0$ with real coefficients can be factored as a product of polynomials of degree 1 or 2 with real coefficients.

**Solution**

We prove it by induction on $n$. When $n=1$ $P(z)=a_1z+a_0$ has degree 1 and there is nothing to prove. This verifies the base of induction.

Suppose we have proved the result for all polynomials of degree $\leq n$. Let $P(z)=a_{n+1}z^{n+1}+a_nz^n+\ldots+a_1z+a_0$ where all $a_i$ are real and $a_{n+1}\neq 0$. By the fundamental theorem of Algebra $P(z)=a_n(z-z_1)(z-z_2)\ldots(z-z_{n+1})$ has a complex root $z_1$ and hence $P(z)=(z-z_1)Q(z)$ where $Q(z)=a_n(z-z_2)\ldots(z-z_{n+1})$ is polynomial of degree $n$.

If $z_1$ is real then the division procedure for polynomials shows that $Q(z)=\frac{P(z)}{z-z_1}$ has real coefficients. Therefore, by the induction assumption it's a product of real polynomials of degree 1 or 2 and hence so is $P(z)$.

Now suppose $z_1$ is not real. Then $z_1=a+bi$ where $a,b\in\mathbb{R}$ and $b\neq 0$. Then by a problem from the previous homework, $\bar{z}_1=a-bi$ is also a root of $P(z)$. WLOG, $z_2=\bar{z}_1=a-bi$.

Then, $P(z)=(z-z_1)(z-z_2)R(z)$ where $R(z)=a_n(z-z_3)\ldots(z-z_{n+1})$ is polynomial of degree $n-1$.

Notice that $P_1(z)=(z-z_1)(z-z_2)=(z-a-bi)(z-a+bi)=(z-a)^2-(bi)^2=z^2-2az+a^2+b^2$ is a quadratic polynomial with real coefficients. Therefore $R(z)=\frac{P(z)}{P_1(z)}$ also has real coefficients. Since it has degree $n-1$, it's a product of real polynomials of degree 1 or 2 and hence so is $P(z)$ $\square$.