(1) Finish the proof of the Cantor-Berenstein Theorem from class.

Recall that the proof goes as follows. Let $f\colon S \to T$ and $g\colon T \to S$ be 1-1. Define S_S to be the set of points in S whose last ancestor is in S. Define S_T to be the set of points whose last ancestor is in T. Define S_∞ to be the set of points in S which have infinitely many ancestors. Define T_S, T_T, T_∞ similarly.

Then define $h\colon S \to T$ by the formula

$$h(s) = \begin{cases} f(x) \text{ if } s \in S_S \cup S_\infty \\ g^{-1}(s) \text{ if } s \in S_T \end{cases}$$

It was shown in class that $h\colon S_S \to T_S$ is 1-1 and onto.

Finish the proof by showing that $h\colon S_T \to T_T$ and $h\colon S_\infty \to T_\infty$ are also 1-1 and onto.

(2) Let $S = (0,1)$ and $T = [0,1)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = \frac{x+1}{2}$.

(a) Find $S_S, S_T, S_\infty, T_S, T_T, T_\infty$

(b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Cantor-Berenstein theorem.

(3) Let $S = P(\mathbb{N})$

Show that $||\mathbb{R}| \le |S|$.

Hint: Since $|\mathbb{R}| = |(0,1)|$ it's enough to show $|(0,1)| \le |S|$. Take a number $x \in (0,1)$, look at its decimal expression $x = 0.a_1a_2a_3\ldots$ and take a subset of N given by numbers whose decimal expressions are $a_1, a_1a_2, a_1a_2a_3, \ldots$.

(4) (a) Let T be an infinite set and let S be a countable set.

Prove that $|T \cup S| = |T|$

(b) Let T be the set of all transcendental numbers.

Prove that $|T| = |\mathbb{R}|$.

Hint: use part a).

(c) Let S be infinite and $A \subset S$ be finite.

Prove that $|S| = |S \backslash A|$.

(5) Find the cardinality of the circle $\{x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.

(6) Find the cardinally of the set of all functions $f\colon \{a,b,c,d\} \to \mathbb{Z}$

(7) Find the cardinally of the set of all functions $f\colon \mathbb{Z} \to \mathbb{Z}$