(1) Let $S = (0,1)$ and $T = [0,1)$. Let $f\colon S \to T$ be given by $f(x) = x$ and $g\colon T \to S$ be given by $g(x) = \frac{x+1}{2}$.

(a) Find $S_S, S_T, S_\infty, T_S, T_T, T_\infty$

(b) give an explicit formula for a 1-1 and onto map $h\colon S \to T$ coming from f and g using the proof of the Cantor-Berenstein theorem.

Solution

Observe that $f(S) = (0,1)$ and $g(T) = [1/2, 1)$. Therefore any $x \in S$ with $0 < x < 1/2$ has no ancestors and thus is its own last ancestor. Therefore, $x \in S_S$. Next, $x = 1/2 = g(0)$ by 0 is not equal to $f(x)$ for any x. therefore, $x = 1/2 \in S_T$.

Next, for $1/2 < x < 3/4$ we see that $x = g(y)$ where $0 < y < 1/2$ and $y = f(y)$. As already observed this y as an element of S has no ancestors and therefore is the last ancestor of x. hence $x \in S_S$.

Next, $x = 3/4 = g(1/2), 1/2 = f(1/2), 1/2 = g(0)$ and 0 is not in the image of S. Thus $0 \in T$ is the last ancestor of $3/4$ and hence $3/4 \in S_T$. proceeding by induction on n we see that all $x \in S$ satisfying $1 - \frac{1}{2^n} < x < 1 - \frac{1}{2^{n+1}}$ has the last ancestor in S and hence $x \in S_S$. Also by induction we see that the last ancestor of $1 - \frac{1}{2^n}$ is $0 \in T$ and hence $1 - \frac{1}{2^n} \in S_T$ for all $n \geq 1$.

This means that $S_T = \{1 - \frac{1}{2^n}, n \geq 1\} = \{1/2, 3/4, 7/8, 15/16. \ldots\}$, $S_S = (0,1) \backslash S_T = \{x \in (0,1) |$ such that $x \neq 1 - \frac{1}{2^n}, n \geq 1\}$ and $S_\infty = \emptyset$.

This implies that $T_\infty = \emptyset$, $T_T = g^{-1}(S_T) = \{1 - \frac{1}{2^n}, n \geq 1\} = \{1 - \frac{1}{2^n}, n \geq 0\} = \{1/2, 3/4, 7/8, 15/16. \ldots\}$ and $T_S = T \backslash T_T$.

Finally, by the proof of the Cantor-Berenstein theorem, the following map $h\colon S \to T$ is 1-1 and onto.

$$h(x) = \begin{cases} f(x) \text{ if } x \in S_S \text{ or } x \in S_\infty \\ g^{-1}(x) \text{ if } x \in S_T \end{cases}$$

In our case this gives

$$h(x) = \begin{cases} x \text{ if } x \in \neq 1 - \frac{1}{2^n} \\ 1 - \frac{1}{2^{n-1}} \text{ if } x = 1 - \frac{1}{2^n} \end{cases}$$

(2) Find the cardinality of the circle $S^1 = \{x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.

Solution

S^1 is a subset of $\mathbb{R}^2$ and therefore $|S^1| \leq |\mathbb{R}^2| = |\mathbb{R}|$.

On the other hand consider the map $f\colon [-1,1] \to S^1$ given by $f(x) = (x, \sqrt{1 - x^2})$. Obviously f is 1-1 and therefore $|[-1,1]| \leq |S^1|$.

We know that $|[-1,1]| = |\mathbb{R}|$ and hence $|\mathbb{R}| = |[-1,1]| \leq |S^1|$.

Thus, by the Cantor-Berenstein theorem, $|S^1| = |\mathbb{R}^2| = |\mathbb{R}| = c$. (recall that the cardinality of $\mathbb{R}$ is denoted by c.)

Answer: $|S^1| = c$.

(3) Find the cardinally of the set of all functions $f\colon \{a, b, c, d\} \to \mathbb{Z}$

Solution

A function $f\colon \{a, b, c, d\} \to \mathbb{Z}$ is determined by the quadruple $(f(a), f(b), f(c), f(d))$ which is an element of $\mathbb{Z}^4$. This gives a 1-1 and onto map from the set of functions $\{f\colon \{a, b, c, d\} \to \mathbb{Z}\}$ to $\mathbb{Z}^4$.

Therefore, $|\{f\colon \{a, b, c, d\} \to \mathbb{Z}\}| = |\mathbb{Z}^4| = |N|$.

Answer: $|\{f\colon \{a, b, c, d\} \to \mathbb{Z}\}| = |\mathbb{Z}^4| = |N|$.

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Find the cardinality of the set $S = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{Q}\}$

Solution

First, observe that $S \subset \mathbb{R}^2$ and hence $|S| \leq |\mathbb{R}^2| = |\mathbb{R}| = c$.

On the other hand, S contains the set $\mathbb{R} \times \{0\} = \{(x, 0) : x \in \mathbb{R}$. Hence $|S| \geq |\mathbb{R} \times \{0\}| = |\mathbb{R}|$. therefore, By Cantor-Beresntein theorem, $|S| = |\mathbb{R}| = c$.

Answer: $|S| = c$.