(1) Let $S = P(\mathbb{N})$

Show that $|\mathbb{R}| \le |S|$.

Hint: Since $|\mathbb{R}| = |(0,1)|$ it's enough to show $|(0,1)| \le |S|$. Take a number $x \in (0,1)$, look at its decimal expression $x = 0.a_1a_2a_3\dots$ and take a subset of N given by numbers whose decimal expressions are $1a_1, 1a_1a_2, 1a_1a_2a_3, \dots$.

(2) (a) Find the cardinally of the set of all functions $f\colon \mathbb{Z} \to \mathbb{Z}$

(b) Let T be an infinite set and let S be a countable set.
Prove that $|T \cup S| = |T|$

(c) Let T be the set of all transcendental numbers.
Prove that $|T| = |\mathbb{R}|$.
Hint: use part b).

(d) Let S be infinite and $A \subset S$ be finite.
Prove that $|S| = |S \backslash A|$.

(3) Which of the following is a field?

(a) the set of all nonnegative rational numbers;

(b) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3}$ where $a, b, c \in \mathbb{Q}$;

(c) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a, b, c, d \in \mathbb{Q}$;

(d) The set of irrational numbers.

(4) Let F be the field consisting of real numbers of the form $p+q\sqrt{2+\sqrt{2}}$ where p, q are of the form $a + b\sqrt{2}$, with a, b rational. Represent

$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}}$$

in this form.

(5) Let t be a transcendental number. Prove that the set $\{(a + bt) : a, b \in \mathbb{Q}\}$ is not a field.

(6) Describe an explicit sequence of ruler and compass operations constructing $\sqrt{\frac{2}{3}}$.