(1) Let $S = P(\mathbb{N})$

Show that $|\mathbb{R}| \le |S|$.

Hint: Since $|\mathbb{R}| = |(0,1)|$ it's enough to show $|(0,1)| \le |S|$. Take a number $x \in (0,1)$, look at its decimal expression $x = 0.a_1a_2a_3\ldots$ and take a subset of N given by numbers whose decimal expressions are $1a_1, 1a_1a_2, 1a_1a_2a_3, \ldots$.

Solution

Consider the following map $f\colon (0,1) \to P(\mathbb{N})$. for $x = 0.a_1a_2a_3\ldots$ set

$$f(x) = \{1a_1, 1a_1a_2, 1a_1a_2a_3, \ldots\}$$

For example, if $x = .2 = .2000\ldots$ then $f(x) = \{12, 120, 1200, 12000, \ldots\}$
By construction f is 1-1 and hence $c = |\mathbb{R}| = |(0,1)| \le |P(\mathbb{N})|$. $\square$

Note that it was proved in class that $|P(\mathbb{N})| \le |\mathbb{R}| = c$. Thus, by Cantor-Berenstein theorem we can further conclude that $|P(\mathbb{N})| = |\mathbb{R}| = c$.

(2) (a) Find the cardinally of the set of all functions $f\colon \mathbb{Z} \to \mathbb{Z}$

Solution

We claim that $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| = |\mathbb{R}| = c$. We will prove that $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| \ge c$ and $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| \le c$. By Cantor-Berenstein theorem this will imply that $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| = c$.
First note that $\{f\colon \mathbb{Z} \to \mathbb{Z}\} \supset \{f\colon \mathbb{Z} \to \{0,1\}\}$ and therefore, $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| \ge |\{f\colon \mathbb{Z} \to \{0,1\}\}|$.
Next, recall that for any set S we have
$|P(S)| = |\{f\colon S \to \{0,1\}\}|$. Thus, $|\{f\colon \mathbb{Z} \to \{0,1\}\}| = |P(\mathbb{Z})|$.
Since $|\mathbb{Z}| = |\mathbb{N}|$ and $|P(\mathbb{N})| = c$ by problem , this implies that $|\{f\colon \mathbb{Z} \to \mathbb{Z}\}| \ge |\{f\colon \mathbb{Z} \to \{0,1\}\}| = |P(\mathbb{Z})| = c$.
To get the opposite inequality consider the following map $G\colon \{f\colon \mathbb{Z} \to \{0,1\}\} \to P(\mathbb{Z} \times \mathbb{Z})$.
Set $G(f) = \Gamma_f$ where Γ_f is the graph of f, i.e. the set $\{(n, f(n))|$ where $n \in \mathbb{Z}\}$. Since different functions have distinct graphs this map is 1-1 and hence $|\{f\colon \mathbb{Z} \to \{0,1\}\}| \le |P(\mathbb{Z} \times \mathbb{Z})| = |P(\mathbb{N})| = c$ where we used that $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$.
Thus $|\{f\colon \mathbb{Z} \to \{0,1\}\}| \le c$ and $|\{f\colon \mathbb{Z} \to \{0,1\}\}| \ge c$ and hence $|\{f\colon \mathbb{Z} \to \{0,1\}\}| = c$.
Answer: $|\{f\colon \mathbb{Z} \to \{0,1\}\}| = c$.

(b) Let T be an infinite set and let S be a countable set.
Prove that $|T \cup S| = |T|$

Solution

First observe that without loss of generality we can assume that $S \cap T = \emptyset$. Indeed, we can always change S to $S_1 = S \backslash T$. It's still countable, $S_1 \cap T = \emptyset$ and $S \cup T = S_1 \cup T$. From now on we will assume $S \cap T = \emptyset$.

Since T is infinite we can find a sequence of **distinct** elements $t_1, t_2, t_3, \ldots$ in T. Let $A = \{t_1, t_2, t_3, \ldots\}$. Then $|A| = |N|$ and A is countable.
By a theorem from class a countable union of countable sets is countable and hence $A \cup S$ is countable since both A and S are. Therefore, $|A \cup S| \leq |N|$. On the other hand, $|N| = |A| \leq |A \cup S|$ and hence $|A \cup S| = |A| = |\mathbb{N}|$. Therefore, there exists $f\colon A \to A \cup S$ which is 1-1 and onto.
Now set $h\colon T \to T \cup S$ by the formula

$$h(x) = \begin{cases} x \text{ if } x \in T \backslash A \\ f(x) \text{ if } x \in A \end{cases}$$

By construction h is 1-1 and onto and hence $|T \cup S| = |T|$. $\square$.

(c) Let T be the set of all transcendental numbers.
Prove that $|T| = |\mathbb{R}|$.
Hint: use part b).

Solution

Let S be the set of algebraic numbers. By a theorem from class S is countable. By definition $T \cup S = \mathbb{R}$.
Next, observe that T is infinite. Suppose not. Then T is finite. Then $\mathbb{R} = T \cup S$ is a union of two countable sets and hence is also countable. This is false. Therefore, T is infinite. Therefore by part b) $|\mathbb{R}| = |T \cup S| = |T|$. $\square$.

(d) Let S be infinite and $A \subset S$ be finite. Prove that $|S| = |S \backslash A|$.

Solution

Let $T = S \backslash A$. Then T is infinite. (Otherwise $S = T \cup A$ is a union of two finite sets and hence is finite.)
Therefore, by part b), $|S| = |T \cup A| = |T|$. $\square$.

(3) Which of the following is a field?
   (a) the set of all nonnegative rational numbers;
   (b) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3}$ where $a, b, c \in \mathbb{Q}$;
   (c) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a, b, c, d \in \mathbb{Q}$;
   (d) The set of irrational numbers.

Solution

(a) the set of all nonnegative rational numbers is NOT a field because it's not close under substraction. $1, 2$ are nonnegative rationals but $1 - 2 = -1$ is not.
(b) Let $F = \{a + b\sqrt{2} + c\sqrt{3}$ where $a, b, c \in \mathbb{Q}\}$. Then F is not a field because it's not closed under multiplication. Indeed, $\sqrt{2}$ and $\sqrt{3}$ are in F but $\sqrt{2} \cdot \sqrt{3}$ is not in f (why?).
(c) the set of numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a, b, c, d \in \mathbb{Q}$; This set is a field. It's equal to F_2 for the tower of fields $F_0 = \mathbb{Q} \subset F_1 = Q(\sqrt{2}) \subset F_2 = F_1(\sqrt{3})$.

(d) The set of irrational numbers is not a field because it does not contain 1.

(4) Let F be the field consisting of real numbers of the form $p+q\sqrt{2+\sqrt{2}}$ where p, q are of the form $a+b\sqrt{2}$, with a, b rational. Represent

$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}}$$

in this form.

Solution

$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}} = \frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}} \cdot \frac{2+3\sqrt{2+\sqrt{2}}}{2+3\sqrt{2+\sqrt{2}}} =$$

$$= \frac{2+5\sqrt{2+\sqrt{2}}+3(2+\sqrt{2})}{4-9(2+\sqrt{2})} = -\frac{8+3\sqrt{2}+5\sqrt{2+\sqrt{2}}}{14+9\sqrt{2}} =$$

$$= -\frac{8+3\sqrt{2}+5\sqrt{2+\sqrt{2}}}{14+9\sqrt{2}} \cdot \frac{14-9\sqrt{2}}{14-9\sqrt{2}} = -\frac{112+42\sqrt{2}+70\sqrt{2+\sqrt{2}}-36\sqrt{2}-18-45\sqrt{2}\sqrt{2+\sqrt{2}}}{196-2\cdot 81}$$

$$= -\frac{96+6\sqrt{2}+(70-45\sqrt{2})\sqrt{2+\sqrt{2}}}{34}$$

(5) Let t be a transcendental number. Prove that the set $\{(a+bt) : a, b \in \mathbb{Q}\}$ is not a field.

Solution

Suppose $F = \{(a+bt) : a, b \in \mathbb{Q}\}$ is a field. Then $t = 0 + 1 \cdot t \in F$ and therefore $t^2 = t \cdot t$ must be in F too. That means that there exist rational a, b such that $t^2 = a + bt$, i.e. $t^2 - bt - a = 0$. Thus t is a root of a quadratic polynomial with rational coefficients and hence is algebraic. This is a contradiction and therefore F is not a field.