(1) Let A be a rectangle in R^n and $Q_1, Q_2, \dots$ be a sequence of closed rectangles such that $A \subset \cup_i Q_i$.

Show that

$$\text{vol}A \leq \sum_{i=1}^{\infty} \text{vol}Q_i$$

(2) Let $f\colon R^n \to R$ be C^1. The graph of f is the set $\Gamma_f = \{(x, y) \in R^{n+1} |$ such that $y = f(x)\}$.

Show that Γ_f has measure zero.

Hint: Show for any rectangle $A \subset R^n$, $f|_A$ is Lipschitz i.e there exists L such that for any $x, y \in A$, $|f(x) - f(y)| \leq L|x - y|$.

(3) Let $A \subset R^m$ be a rectangle.

a function $f = (f_1, \dots, f_m)\colon A \to R^m$ is called integrable over A if each f_i is integrable over A. In this case we define

$\int_A f := (\int_A f_1, \dots, \int_A f_m)$.

Show that if f is integrable over A then $|f|$ is also integrable over A. Is the converse always true?

(4) Let $f_i\colon A \to R$ be a sequence of functions integrable over A. Suppose $f_i \to f$ *uniformly* on A as $i \to \infty$. Recall that this means that for any $\epsilon > 0$ there exists N such that for all $x \in A$ and all $i > N$ we have $|f_i(x) - f(x)| \leq \epsilon$.

Prove that f is integrable over A.

Hint: Show that if $|f_i(x) - f(x)| \leq \epsilon$ then $|U(f, P) - U(f_i, P)| \leq \epsilon$ for any partition of A.

Extra Credit. To be written up and handed in separately from the rest of the homework.

Give an example of a diffeomorphism between open sets in R^n which is not C^1.

Hint: Look at the map $f\colon R \to R$ given by

$$f(x) = \begin{cases} 3x + x^2 \sin(1/x) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$