(1) Give a careful proof of the following fact mentioned in class. Let M be a manifold with boundary in R^n. Let $p \in \partial M$. Let $n \in T_p(M)$ be a vector normal to $T_p\partial M$. let $f\colon W \to R^n$ be a parametrization of M near p (here W is an open set in R^k) such that $f(W \cap R^k_+) = M \cap U$ and $f(W \cap R^{k-1} \times \{0\}) = \partial M \cap U$. Let $p = f(q)$. Suppose n is "outward" with respect to f, i.e. $n = df_q(v)$ and $v = (v_1, \ldots, v_k)$ with $v_k < 0$.

Prove that n is "outward" with respect to any other parametrization of M near p.

(2) Let M^k be a manifold in R^n. Show that for every $p \in M$ there exists an open set $U \subset R^n$ containing p such that $U \cap M$ admits smooth vector fields $X_1, \ldots, X_k$ tangent to M such that for every $x \in U \cap M$, the vectors $X_1(x), \ldots, X_k(x)$ are orthonormal.

(3) Let $M = S^2 = \{x^2 + y^2 + z^2 = 1\}$ in R^3. Let $f\colon R^3 \to R$ be given by $f(x, y, z) = xy + z^2$. Find the minimum and the maximum of F on M.

(4) Let $M = \{x^2 + y^2 + z^2 \leq 1\}$ in R^3. Consider the induced orientation on ∂M and find a positive basis of $T_p\partial M$ at $p = (1, 0, 0)$.

Further, let $N = S^2_+ = \{(x, y, z)|$ such that $x^2 + y^2 + z^2 = 1$ and $z \geq 0\}$. Consider the orientation on N coinciding with the orientation on $S^2 = \partial M$. Consider ∂N with the induced orientation from N. Find a positive basis of $T_p\partial N$ for $p = (1, 0, 0)$.

(5) Let M be the cylinder $\{(x, y, z)|$ such that $x^2 + y^2 = 1$ and $0 \leq z \leq 1\}$ in R^3. Let $\omega = z dx$. Fix an orientation on M such that the $e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ give a positive basis of T_pM for $p = (1, 0, 0)$.

Compute $\int_M d\omega$ and $\int_{\partial M} \omega$ and verify that they are equal.