(1) Let $f(r,\theta) = (r\cos\theta, r\sin\theta)$ be a map $f\colon U \to R^2$ where $U = \{(r.\theta)|r > 0, -\pi/2 < \theta < \pi/2\}$.

Let $g(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$ be a map $g\colon V \to R^2$ where $V = \{(x,y)|x > 0\}$.

Show that $f\colon U \to V$ is a bijection with $f^{-1} = g$. Compute $df(r,\theta), dg(x,y)$ and verify that $[dg(f(r,\theta)] = [df(r,\theta)]^{-1}$.

(2) A set $A \subset R^n$ is called *convex* if for any points $x, y \in A$, the straight line segment from x to y is contained in A.

Prove that the ball $B(0,1) = \{x \in R^n|$ such that $|x| < 1\}$ is convex.

Hint: Parametrize the line segment from x to y by $z(t) = x + t(y-x)$ where $0 \le t \le 1$ and prove that $\sqrt{< z(t), z(t) >} = |z(t)| < 1$ for any $t \in [0,1]$.

(3) A map $h\colon R^n \to R^n$ is called a *diffeomorphism* if h is 1-1 and onto and both h and h^{-1} are differentiable.

Let $U \subset R^n$ be open and $f\colon U \to R^n$ be any map. let $h\colon R^n \to R^n$ be a diffeomorphism.

(a) Let $T\colon R^n \to R^n$ be an invertible linear map. Show that T is a diffeomorphism;
(b) f is continuous if and only if $h \circ f$ is continuous;
(c) f is differentiable if and only if $h \circ f$ is differentiable;
(d) f is 1-1 if and only if $h \circ f$ is 1-1;
(e) $f(U)$ is open if and only if $(h \circ f)(U)$ is open;
(f) Let $f\colon U \to f(U)$ be 1-1 and $f(U)$ is open.
Show that f^{-1} is differentiable if and only if $(h \circ f)^{-1}$ is differentiable.

(4) Let $f\colon R^n \to R^n$ be C^1, 1-1 and onto. Assume $df(x)$ is invertible for any $x \in R^n$. Prove that f^{-1} is C^1.

(5) Let $f\colon R^3 \to R$ be given by $f(x,y,z) = \sin(xyz) + e^{2x+y(z-1)}$. show that the level set $\{f = 1\}$ can be solved as $x = x(y,z)$ near $(0,0,0)$ and compute $\frac{\partial x}{\partial y}(0,0)$ and $\frac{\partial x}{\partial z}(0,0)$

(6) let $f\colon R^3 \to R^2$ be given by $f_1(x,y,z) = \sin(x+y) - x + 2z$, $f_2(x,y,z) = y + \sin z$ Show that the level set $\{f_1 = 0, f_2 = 0\}$ can be solved near $(0,0,0)$ as $y = y(x), z = z(x)$ and compute $\frac{\partial y}{\partial x}(0)$ and $\frac{\partial z}{\partial x}(0)$

Extra Credit: Let $U \subset R^n$ be open and $f\colon U \to R^m$ be C^1 where $m < n$. prove that f can not be 1-1 on U.