**MAT 257Y** **Practice Final**

(1) Let $A \subset R^n$ be a rectangle and let $f\colon A \to R$ be bounded. Let $P_1, P_2$ be two partitions of $A$ . Prove that $L(f, P_1) \leq U(f, P_2)$.

(2) let $M = \{(x, y) \in R^2|$ such that $x^2 + y^2 = 1$. let $f\colon M \to R$ be given by $f(x, y) = x^2 + y$. Find the minimum and the maximum of $f$ on $M$.

(3) Let $T\colon R^{2n} = R^n \times R^n \to R$ be a 2-tensor on $R^n$. Show that $T$ is differentiable at $(0, 0)$ and compute $df(0, 0)$.

(4) Let $\omega = \frac{xdy\wedge dz+ydz\wedge dx+zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form on $R^3\backslash(0, 0, 0)$.

Verify that $\omega$ is closed.

*Hint:* One way to simplify the computation is to write $\omega = f \cdot \tilde{\omega}$ where $f = \frac{1}{(x^2+y^2+z^2)^{3/2}}$ and $\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx$.

(5) Let $f\colon R^2 \to R^2$ be given by $f(x, y) = (e^{2y}, 2x + y))$ and let $\omega = x^2ydx + ydy$.

Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

(6) Determine if $\int_{0<x^2+y^2<1}^{ext} \ln(x^2+y^2)$ exists and if it does compute it.

(7) Let $U, V$ be open in $R^n$. Let $f\colon R^n \to R$ be a continuous nonnegative function such that $\int_U^{ext} f$ and $\int_V^{ext} f$ exist.

Prove that $\int_{U\cup V}^{ext} f$ exists.

*Hint:* use compact exhaustions of $U$ and $V$ to construct a compact exhaustion of $U \cup V$.

(8) Let $F(x) = \int_{e^x}^{x^2} f(tx)dt$ where $f\colon R \to R$ is $C^1$.

Show that $F(x)$ is $C^1$ and find the formula for $F'(x)$.

(9) Prove that a compact set is closed.

(10) Let $x(t_1, t_2) = t_1 \cos t_2, y(t_1, t_2) = t_1^2 + e^{t_1t_2}$. Let $f(x, y)$ be a differentiable function $f\colon R^2 \to R$. Let

$g(t_1, t_2) = f(x(t_1, t_2), y(t_1, t_2))$. Express $\frac{\partial g}{\partial t_1}(1, 0)$ and $\frac{\partial g}{\partial t_2}(1, 0)$ in terms of partial derivatives of $f$.

(11) Mark true or false. Justify your answer.

Let $A, B$ be any subsets of $R^n$.

(a) $br(A) \subset Lim(A)$

(b) $Lim(A) \subset A$

(c) $br(A \cap B) \subset br(A) \cap br(B)$.

(12) Let $M^3$ be a compact 3-manifold with boundary in $R^3$ and let $n$ be the outward unit normal on $\partial M$. Let $F = (F_1, F_2, F_3)$ be a vector field on $R^3$. Prove that

$$\int_M divF = \int_{\partial M} \langle F, n \rangle$$

Convert the integral over $\partial M$ to an integral of a form in $R^3$ and use Stokes' formula.

(13) let $M^2 \subset R^3$ be the torus of revolution obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the $xz$ plane around the $yz$ axis. Consider the orientation on $M$ induced by the outward normal field $N$ where $N(3, 0, 0) = (1, 0, 0)$.

Find $\int_M x dy \wedge dz$

(14) Let $M \subset R^n$ be an oriented manifold.

Prove that $\text{vol}(M) = \int_M dA$ is positive.