MAT 257Y **Practice Term Test 3**

(1) Let A be a Jordan measurable set.

prove that for any $\epsilon > 0$ there exists a compact Jordan measurable set $C \subset U$ such that $\int_{A\setminus C} 1 < \epsilon$.

Hint: Consider the lower Riemann sum for $\int_A 1$.

Solution

Let P be a partition such that $\int_A 1 - L(f, P) < \epsilon$. Then

$L(f, P) = \sum_{Q\in P} m_Q 1\text{vol}(Q) = \sum_{Q\in P, Q\subset A} 1\cdot\text{vol}(Q) = \int_C 1$ where $C = \cup_{Q\in P, Q\subset A} Q$.

This means that $\int_A 1 - \int_C 1 = \int_{A\setminus C} 1 < \epsilon$.

(2) Let ϕ_i be a partition of unity on an open set U. let $K \subset U$ be a compact set.

Prove that all but finitely many ϕ_i vanish on K.

Solution

By definition of a partition of unity, for every point $p \in K$ there exists $\epsilon_p > 0$ such that all but finitely many ϕ_i vanish on $B(p, \epsilon_p)$.

We have that $\cup_{p\in K} B(p, \epsilon_p) \supset K$. By compactness of K we can choose a finite cover of K by the balls $B(p_i, \epsilon_i)$ and the result follows.

(3) Let $c\colon [0, 1] \to (R^n)^n$ be continuous. Suppose that $c^1(t), \ldots, c^n(t)$ is a basis of R^n for any t.

Prove that $(c^1(0), \ldots, c^n(0))$ and $(c^1(1), \ldots, c^n(1))$ have the same orientation.

Solution

Let $f(t) = \det[c^1(t), \ldots, c^n(t)]$. Then $f(t)$ is continuous and never zero. therefore $f(t) > 0$ for all t or $f(t) < 0$ for all t by the intermediate value theorem. In either case $f(1)/f(0) > 0$. Let A be the transition matrix from $(c^1(0), \ldots, c^n(0))$ to $(c^1(1), \ldots, c^n(1))$. then $A = [c^1(0), \ldots, c^n(0)]^{-1}[c^1(1), \ldots, c^n(1)]$. hence

$\det(A) = f(1)/f(0) > 0$ which means that $(c^1(0), \ldots, c^n(0))$ and $(c^1(1), \ldots, c^n(1))$ have the same orientation.

(4) Let C be the triangle in R^2 with vertices $(0,0), (1,2), (-1,3)$

Compute $\int_C x + y$.

Hint: use a linear change of variables.

Solution

Let's make a change of variable

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

or $x = u - v, y = 2u + 3v$. we have that $\det \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} =$ 5. Therefore $\int_C x + y = \int_U 5((u-v)+(2u+3v))$ where $U = \{(u,v) | u > 0, v > 0, u + v < 1\}$. Therefore using Fubini's theorem we compute

$$\int_U 5((u-v) + (2u+3v)) = \int_0^1 \int_0^{1-u} 5(3u+2v) dv du =$$

$$= 5 \int_0^1 (3uv + v^2)|_0^{1-u} du = 5 \int_0^1 3u(1-u) + (1-u)^2 du =$$

$$= 5 \int_0^1 -2u^2 + u + 1 du = 5(-2/3u^3 + u^2/2 + u)|_0^1 = 25/6$$

(5) Let $T \in \mathcal{T}^2(V)$.

Prove that $Alt(T) = 0$ if and only if T is symmetric. Is the same true if $T \in \mathcal{T}^3(V)$?

Solution

If $T \in \mathcal{T}^2(V)$ then by definition $Alt(T)(u,v) = \frac{1}{2}(T(u,v) - T(v,u))$. Thus $Alt(T) = 0$ mean $T(u,v) = T(v,u)$ for any $u, v \in V$, i.e. T is symmetric.

The same is false if $k > 2$. For example $T = e_1^* \otimes e_2^* \otimes e_3^* + 2e_2^* \otimes e_1^* \otimes e_3^* + e_2^* \otimes e_3^* \otimes e_1^*$ is not symmetric. However, $Alt(T) = \frac{1}{6}(e_1^* \wedge e_2^* \wedge e_3^* - 2e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_2^* \wedge e_3^*) = 0$.

(6) Let $T \subset \mathcal{T}^2(V)$. Let T_{ij} be coordinates of T with respect to basis $e_1, \ldots, e_n$ and $\tilde{T}_{ij}$ be coordinates of T with respect to basis $\tilde{e}_1, \ldots, \tilde{e}_n$. Let A be the transition matrix from e to $\tilde{e}$.

Prove that $[\tilde{T}] = A^t[T]A$.

Solution

We have $\tilde{e} = e \cdot A$ so that $\tilde{e}_i = \sum_j e_j A_{ji}$. Therefore $\tilde{T}_{ij} = T(\tilde{e}_i, \tilde{e}_j) = T(\sum_k e_k A_{ki}, \sum_l e_l A_{lj}) = \sum_{k,l} A_{ki} T(e_k, e_l) A_{lj} = \sum_{k,l} A^t_{ik} T_{kl} A_{lj} = (A^t T A)_{ij}$

(7) Let $f\colon R^n \to R^n$ be a C^∞ diffeomorphism. Let $\omega = dx^1 \wedge \ldots \wedge dx^n$. Suppose $f^*\omega = \omega$.

Prove that vol$U$ = vol$f(U)$ for any bounded open set U.

Solution

by a theorem from class $f^*\omega = \det[df]\omega$. this means that $\det[df] = 1$ and the statement follows by the change of variables theorem.

(8) Let $f\colon R^2 \to R^3$ be given by $f(x, y) = (x^2 + \cos(xy), e^{2xy}, xy^2)$. let $\omega = e^{xy} dx \wedge dz + 2x dy \wedge dz - \sin(xy) dy \wedge dz$

Compute $f^*\omega$.

Solution

$f^*\omega = e^{(x^2+\cos(xy))(e^{2xy})} d(x^2 + \cos(xy)) \wedge d(xy^2) + [2(x^2 + \cos(xy)) - \sin((x^2 + \cos(xy))e^{2xy})] d(e^{2xy}) \wedge d(xy^2) = e^{(x^2+\cos(xy))(e^{2xy})}((2x - y\sin(xy))dx - x\sin(xy)dy) \wedge (y^2 dx + 2xy dy) + [2(x^2 + \cos(xy)) - \sin((x^2 + \cos(xy))e^{2xy})](2ye^{2xy} dx + 2xe^{2xy} dy) \wedge (y^2 dx + 2xy dy) = e^{(x^2+\cos(xy))(e^{2xy})}((2x - y\sin(xy))2xy + x\sin(xy)y^2) dx \wedge dy + [2(x^2 + \cos(xy)) - \sin((x^2 + \cos(xy))e^{2xy})][2ye^{2xy}2xy - 2xe^{2xy}y^2] dx \wedge dy$