Solution to Term Test 2

(1) (15 pts) Let $f = (f_1, f_2)\colon R^3 \to R^2$ be a C^1 map such that $f(0,0,0) = (0,0)$, $[df_1(0,0,0)] = [1,0,2]$, $[df_2(0,0,0)] = [-1,1,1]$. Denote the standard coordinates in R^3 by (x_1, x_2, y). Further, denote (x_1, x_2) by x.

Show that near the point $(0,0,0)$ the level set $\{(x,y) \in R^3 | \quad f(x,y) = 0\}$ can be written as a graph of a differentiable function $x = g(y)$ and find $g'(0)$.

Solution

First we look at the matrix $[\frac{\partial f}{\partial x}(0)] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

This matrix has nonzero determinant hence $g(y)$ exists by Implicit Function Theorem. To compute $g'(0)$ we differentiate the equality $f(g(y), y) = 0$. By the chain rule we get $\frac{\partial f}{\partial x}(0)g'(0) + \frac{\partial f}{\partial y}(0) = 0$ or

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot g'(0) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$$

Solving this linear system we find $g'(0) = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.

(2) (15 pts) Let A be a rectangle in R^n and let $S \subset A$ be closed.

Show that S has content 0 if an only if it has measure 0.

Solution

Suppose S has measure zero. Since it's closed and bounded (as a subset of a rectangle) it is compact. Let $\epsilon > 0$. Then there exists a countable cover of S by rectangles Q_i such that $\sum_{i=1}^{\infty} \mathrm{vol} Q_i < \epsilon/2$. For every Q_i there exists a slightly bigger open rectangle Q_i' such that $Q_i \subset Q_i'$ and $\mathrm{vol} Q_i' < 2\mathrm{vol} Q_i$. By compactness of S we can choose a finite cover $Q_1', \ldots, Q_N'$ such that $S \subset \cup_{i=1}^{N} Q_i'$.

Then $\sum_{i=1}^{N} \mathrm{vol} Q_i' < \sum_{i=1}^{\infty} \mathrm{vol} Q_i' < 2\sum_{i=1}^{\infty} \mathrm{vol} Q_i < 2\epsilon/2 = \epsilon$. Since $\epsilon > 0$ is arbitrary this means that S has content 0.

The other direction is obvious by the definition of a set of measure zero.

(3) (15 pts) Mark True or False. If true, give a proof. If false, give a counterexample.
   (a) If $S \subset R^n$ has content 0 then S is bounded.
   (b) If $S \subset R^n$ has measure 0 then $br(S)$ also has measure 0.
   (c) $br(S_1 \cap S_2) \subset br(S_1) \cap br(S_2)$ for any sets $S_1, S_2 \subset R^n$.

Solution

   (a) **True.** If S has finite content it can be covered by a finite collection of rectangles and hence it is bounded.
   (b) **False.** Counterexample: $S = Q \cap [0,1]$. then S has measure zero but $br(S) = [0,1]$ does not have measure zero.
   (c) **False.** Counterexample: Take S_1 and S_2 to be two intersecting disks in R^2.

(4) (15 pts) Let $f\colon [0,1] \times [0,1] \to R$ be defined by $f(x,y) = x + 2y$. Let P be the partition $\{0, 1/2, 1\} \times \{0, 1/2, 1\}$. Find $L(f, P)$.

Solution

Let $Q_1 = [0,1/2] \times [0,1/2]$, $Q_2 = [0,1/2] \times [1/2,1]$, $Q_3 = [1/2,1] \times [0,1/2]$, $Q_4 = [1/2,1] \times [1/2,1]$. Then $L(f,P) = \sum_{i=1}^{4} m_{Q_i}(f) \mathrm{vol} Q_i = \frac{1}{4}\sum_{i=1}^{4} m_{Q_i}(f)$.

To compute $m_{Q_1}(f)$ we observe that for $0 \le x \le 1/2, 0 \le y \le 1/2$ we have $x + 2y \ge 0 + 2 \cdot 0 = 0 = f(0,0)$ hence $m_{Q_1}(f) = 0$. Similarly, $m_{Q_2}(f) = 0 + 2 \cdot 1/2 = 1 = f(0, 1/2)$, $m_{Q_3}(f) = 1/2 + 2 \cdot 0 =$

$1/2 = f(1/2, 0)$ and $m_{Q_4}(f) = 1/2 + 2 \cdot 1/2 = 3/2 = f(1/2, 1/2)$

Hence $L(f, P) = \frac{1}{4}(0 + 1 + 1/2 + 3/2) = 3/4$.

(5) (15 pts) Let $f\colon [a, b] \to R$ be continuous. Show that the graph of f has content 0. Recall that the graph of f is the set $\Gamma_f = \{(x, y) \in R^2 | \quad x \in [a, b], y = f(x)\}$.

Hint: Use that f is uniformly continuous.

Solution

Solution 1. Since f is continuous on $[a, b]$ and $[a, b]$ is compact, f is uniformly continuous on $[a, b]$. Let $\epsilon > 0$. There exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon/2$. Choose N such that $1/N < \delta$ and let P be the partition given by $x_i = i/n$, $i = 0, \ldots, N$. By uniform continuity for any $x \in [x_{i-1}, x_i]$ we have $|f(x) - f(x_i)| < \epsilon/2$. therefore the graph of f on $[x_{i-1}, x_i]$ is contained in the rectangle $Q_i = [x_{i-1}, x_i] \times [f(x_i) - \epsilon/2, f(x_i) + \epsilon/2]$. Thus the graph of f is covered by Q_is and

$\sum_{i=1}^{N} \mathrm{vol} Q_i = N \cdot (1/N) \cdot \epsilon = \epsilon$

Hence the graph has content 0.

Solution 2. f is continuous on $[a, b]$ and hence it is integrable on $[a, b]$. Let $\epsilon > 0$. then there exists a partition P of $[a, b]$ such that

$U(f, P) - L(f, P) < \epsilon$. We rewrite $U(f, P) - L(f, P) = \sum_{Q \in P}(M_Q(f) - m_Q(f))\mathrm{vol} Q$. For any $Q \in P$ observe that the graph of f over Q is contained in the rectangle $Q \times [m_Q(f), M_Q(f)]$ and $\mathrm{vol}(Q \times [m_Q(f), M_Q(f)]) = (M_Q(f) - m_Q(f))\mathrm{vol} Q$. In other words

$\epsilon > U(f, P) - L(f, P) = \sum_{Q \in P} \mathrm{vol}(Q \times [m_Q(f), M_Q(f)])$ and hence the graph of f has content 0.

(6) (15 pts) Let A be a rectangle in R^n and let $f\colon A \to R$ be integrable over A. Let $c > 0$ be a constant.

Prove that $\int_A cf$ exists and is equal to $c \int_A f$.

Solution

Let P be any partition f A then

$$L(cf,P)=\sum_{Q\in P} m_Q(cf)\mathrm{vol}Q$$

Claim: $m_Q(cf)=cm_Q(f)$ for any Q and any positive c. Indeed, let $m=\inf_{x\in Q} f(x)$. Then $f(x)\geq m$ for any $x\in Q$ and hence $Cf(x)\geq cm$ for any $x\in Q$. This means that $m_Q(cf)\geq cm_Q(f)$. applying the above to cf and $1/c$ we get $m_Q(f)=m_Q(\frac{1}{c}cf)\geq \frac{1}{c}m_Q(cf)$ which means that $m_Q(cf)=m_Q(cf)$.

Hence

$$L(cf,P)=\sum_{Q\in P} m_Q(cf)\mathrm{vol}Q=\sum_{Q\in P} cm_Q(f)\mathrm{vol}Q=cL(f,P)$$

Similarly $U(cf,P)=cU(f,P)$. Therefore

$$\sup_P L(cf,P)=\sup_P cL(f,P)=c\sup L(f,P)=c\int_A f$$

and

$$\inf_P U(cf,P)=\inf_P cU(f,P)=c\inf_P U(f,P)=c\int_A f$$

hence cf is integrable and $\int_A cf=c\int_A f$.

(7) (10 pts) Let $f\colon [-\pi/2,\pi/2]\to R$ be given by $f(x)=\cos x$. Let $S\subset R^3$ be the solid obtained by rotating around the x-axis the region between the graph of f and the x-axis.

Compute volume of S.

Recall that $\cos^2\alpha=\frac{1+\cos 2\alpha}{2}$.

Solution

By definition, $\mathrm{vol}S=\int_S 1$. Using Fubini's theorem, we compute.

$$\mathrm{vol}S = \int_S 1 = \int_{-\pi/2}^{\pi/2} (\int_{S_x} 1) dx$$

where S_x is cross-section of S at level x, i.e it's the circle of radius $\cos x$ centered at 0. We have $\int_{S_x} 1 = area(S_x) = \pi \cos^2 x$. Therefore,

$$\mathrm{vol}S = \int_S 1 = \int_{-\pi/2}^{\pi/2} (\int_{S_x} 1) dx = \int_{-\pi/2}^{\pi/2} \pi \cos^2 x dx = \pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2x}{2} dx =$$

$$= \pi/2(x + \frac{\sin 2x}{2})|_{-\pi/2}^{\pi/2} = \frac{\pi^2}{2}$$