MAT 257Y

- (1) (15 pts) Give the definitions of the following notions.
 - (a) an open set in \mathbb{R}^n ;
 - (b) a boundary point of a set $A \subset \mathbb{R}^n$;
 - (c) a function $f: \mathbb{R}^n \to \mathbb{R}^m$ differentiable at a point p;
 - (d) a directional derivative of a function $f: \mathbb{R}^n \to \mathbb{R}^m$ at a point p.

Solution

- (a) A set $U \subset \mathbb{R}^n$ is called open if for every $p = (p_1, p_2, \dots, p_n) \in U$ there exists $\epsilon > 0$ such that the rectangle $I = (p_1 - \epsilon, p_1 + \epsilon) \times (p_1 - \epsilon, p_2 + \epsilon) \times \dots \times (p_n - \epsilon, p_n + \epsilon)$ is contained in U.
- (b) a point p is called a boundary point of A if for any $\epsilon > 0$ there exist $a \in B(p, \epsilon) \cap A$ and $b \in B(p, \epsilon) \cap A^c$
- (c) a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point p if there exists a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(p+h) - f(p) - T(h)}{|h|} = 0$$

- (d) Let $X \in \mathbb{R}^n$ and let g(t) = f(p + tX). Then $D_X f(p) = g'(0)$ if it exists is called the directional derivative of f at p in the direction X.
- (2) (15 pts) Find the partial derivatives of the following functions
 - (a)

$$f(x,y) = \int_{x}^{\int_{x}^{y} g(t)dt} g(t)dt$$

Hint: put $F(x, y) = \int_x^y g(t)dt$ and express f as a composition.

(b) $f(x,y) = \ln((\sin(x+y^2))^{\cos 2x})$

Solution

(a) put $F(x, y) = \int_x^y g(t) dt$. By the fundamental theorem of calculus we have

 $\frac{\partial F}{\partial x}(x,y)(x,y) = -g(x)$ and $\frac{\partial F}{\partial y}(x,y)(x,y) = g(y)$

We also have that f(x, y) = F(x, F(x, y)). Therefore, by the chain rule we have

$$\begin{split} \frac{\partial f}{\partial x}(x,y) &= \frac{\partial F}{\partial x}(x,F(x,y))\frac{\partial x}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,F(x,y))\frac{\partial F}{\partial x}(x,y) = \\ &= -g(x)\cdot 1 + g(F(x,y))\cdot (-g(x)) = -g(x) - g(\int_x^y g(t)dt)g(x) \\ &\qquad \text{Similarly.} \end{split}$$

y,

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) &= \frac{\partial F}{\partial x}(x,F(x,y))\frac{\partial x}{\partial y}(x,y) + \frac{\partial F}{\partial y}(x,F(x,y))\frac{\partial F}{\partial y}(x,y) = \\ &= -g(x) \cdot 0 + g(F(x,y))g(y) = g(\int_{x}^{y} g(t)dt)g(y) \\ \text{(b) First we simplify } f(x,y) &= \ln((\sin(x+y^2))^{\cos 2x}) = \\ &\quad (\cos 2x)\ln(\sin(x+y^2)) \\ &\text{Then we compute} \\ &\quad \frac{\partial f}{\partial x}(x,y) = -2(\sin 2x)\ln(\sin(x+y^2)) + (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2) \\ &\quad \frac{\partial f}{\partial y}(x,y) = (\cos 2x)\frac{1}{\sin(x+y^2)}\cos(x+y^2)(2y) \\ \text{(3) (20 pts) Let } f: R^2 \to R \text{ be given by the formula} \\ &\quad f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases} \end{aligned}$$

(a) Show that f(x, y) is continuous at (0, 0).

- (b) Show that f has partial derivatives at (0,0).
- (c) Show that f is not differentiable at (0,0).

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Solution

- (a) we rewrite $f(x, y) = y \frac{x^2}{x^2 + y^2}$. Clearly $|\frac{x^2}{x^2 + y^2}| \le 1$ and $\lim_{(x,y)\to(0,0)} y = 0$. therefore $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.
- (b) by definition of f we see that f(x,0) = 0 and f(0,y) = 0. therefore $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.
- (c) Suppose f is differentiable at (0, 0). then the matrix $[df(0, 0)] = [\frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0)] = [0, 0]$. By definition of differentiability this would mean that

$$\lim_{h \to 0} \frac{f(h) - f(0) - 0}{|h|} = 0$$

However along the line (t, t) we have

$$\lim_{t \to 0} \frac{f(t,t) - f(0,0)}{|t|} = \lim_{t \to 0} \frac{\frac{t^3}{2t^2}}{|t|} \neq 0$$

This is a contradiction which means that f is not differentiable at (0,0).

(4) (10 pts) Show that a compact subset of \mathbb{R}^n is bounded.

Solution

Let $C \subset \mathbb{R}^n$ be compact. Let $U_n = B(0,n)$ where n = 1, 2, 3... Then U_n is open and $\bigcup_n U_n = \mathbb{R}^n$. Hence $C \subset \bigcup_n U_n$. By definition of compactness we can choose a finite subcover U_{n_1}, \ldots, U_{n_k} still covering C. Let $m = \max_k n_k$. Then $C \subset U_m$ and hence it is bounded.

(5) (10 pts) let $f(x, y) = x^2 + 5y^2 - 4xy - 2y$. Find all possible points of minimum of f(x, y).

Solution

f is clearly differentiable everywhere. Its minimum can occur only at points where both partial derivatives vanish. we compute

$$\frac{\partial f}{\partial x}(x,y) = 2x - 4y \frac{\partial f}{\partial y}(x,y) = 10y - 4x - 2$$

we solve

$$\begin{cases} 2x - 4y = 0\\ 10y - 4x - 2 = 0 \end{cases} \begin{cases} x = 2\\ y = 1 \end{cases}$$

Thus the only possible point of minimum is (2,1).

(6) (15 pts) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be continuous.

Are the following statements true or false? Prove if true and give counterexamples if false.

- (a) If $A \subset \mathbb{R}^n$ is closed and bounded then f(A) is closed and bounded.
- (b) If $A \subset \mathbb{R}^n$ is closed then f(A) is closed.
- (c) If $A \subset \mathbb{R}^n$ is bounded then f(A) is bounded.

Solution

(a) **True**

If $A \subset \mathbb{R}^n$ is closed and bounded if and only if it's compact and and image of a compact set under a continuous map is compact.

- (b) False. let $f(x) = \arctan x$. Then $f([0, \infty)) = [0, \pi/2)$ is not closed.
- (c) **True**

If A is bounded it's contained in a closed ball $B = \overline{B}(0, R) = \{x \in \mathbb{R}^n \text{ such that } |x| \leq R\}$ for some R > 0. Then $f(A) \subset f(B)$ but B is compact. hence f(B) is also compact and in particular it is bounded.

(7) (15 pts) Let GL(n, R) be the set of all $n \times n$ invertible matrices.

Show that GL(n, R) is open in R^{n^2} .

Solution

Let $f: \mathbb{R}^{n^2} \to \mathbb{R}$ be given by $f(A) = \det(A)$. then f is a polynomial in coordinate entries and hence is continuous. We know that A is invertible if an only if $\det(A) \neq 0$.

Therefore $GL(n, R) = f^{-1}((-\infty, 0) \cup (0, \infty))$ and hence is open as the preimage of an open set.