MAT 257Y **Solutions to Term Test 3**

(1) (15 pts) Give the following definitions

(a) A k-tensor on a vector space V.

(b) Cross product on R^n.

(c) A partition of unity on on open set U subordinate to an open cover U_α.

Solution

(a) A k-tensor on a vector space V is a map $T\colon \overbrace{V\times\ldots\times V}^{k\text{ times}} \to R$ which is linear in every variable, i.e
$T(v_1,\ldots v_{i-1}, av_i'+bv_i'', v_{i+1},\ldots,v_k) = aT(v_1,\ldots v_{i-1}, v_i', v_{i+1},\ldots,v_k)+$
$bT(v_1,\ldots v_{i-1}, v_i'', v_{i+1},\ldots,v_k)$ for any $i=1,\ldots,k$.

(b) Cross product on R^n. For $v_1,\ldots v_{n-1}\in R^n$ we define $v_1\times\ldots\times v_n$ as the unique vector $v\in R^n$ such that $\langle v,w\rangle = \det\begin{pmatrix} v_1\\ \ldots\\ v_{n-1}\\ w\end{pmatrix}$ for any $w\in R^n$.

(c) A partition of unity on an open set U subordinate to an open cover U_α is a sequence of functions $\phi_i\colon R^n\to R$ with the following properties

(i) ϕ_i is C^∞ for any i.

(ii) For any i there exists α such that $supp(\phi_i)\subset U_\alpha$

(iii) $\phi_i\ge 0$ for any i.

(iv) for any $p\in U$ there exists $\epsilon>0$ such that $B(p,\epsilon)$ intersects only finitely many $supp(\phi_i)$.

(v) $\sum_{i=1}^\infty \phi_i = 1$ on U.

(2) (10 pts) Let $U\subset R^n$ be open. Let $f,g\colon U\to R$ be continuous and $|f|\le g$. Suppose $\int_U^{ext} g$ exists.

Prove that $\int_U^{ext} f$ also exists.

Solution

Let ϕ_i be a partition of unity on U. Then by definition of extended integral, $\sum_{i=1}^\infty \int_U |g|\phi_i<\infty$

Therefore

$\sum_{i=1}^\infty \int_U |f|\phi_i \le \sum_{i=1}^\infty \int_U |g|\phi_i<\infty$ and hence $\int_U^{ext} f$ exists by the definition.

(3) (15 pts) Let e_1,e_2,e_3,e_4 be a basis of a 4-dimensional space V.

Let $\omega = Alt(e_1^*\otimes e_2^* + e_3^*\otimes e_4^*)$ and $\eta = 2e_2^*+e_3^*$.

Find $\omega\wedge\eta(e_2,e_3,e_4)$.

Solution

We have $\omega = \frac{1!1!}{2!}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*)$. Hence $\omega \wedge \eta = \frac{1}{2}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) \wedge (2e_2^* + e_3^*) = \frac{1}{2}(e_1^* \wedge e_2^* \wedge 2e_2^* + e_1^* \wedge e_2^* \wedge e_3^* + e_3^* \wedge e_4^* \wedge 2e_2^* + e_3^* \wedge e_4^* \wedge e_3^*) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^* + e_3^* \wedge e_4^* \wedge e_2^* = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^* + e_2^* \wedge e_3^* \wedge e_4^*$.

Therefore $\omega \wedge \eta(e_2, e_3, e_4) = \frac{1}{2}e_1^* \wedge e_2^* \wedge e_3^*(e_2, e_3, e_4) + e_2^* \wedge e_3^* \wedge e_4^*(e_2, e_3, e_4) = 0 + 1 = 1$.

(4) (15 pts) Let $U = \{(x, y) \in R^2|$ such that $0 < x^2 + y^2/4 < 1, y > 0, -y/2 < x < y/2$. Compute $\int_U y$.

Hint: use the appropriate change of variables.

Solution

First we make the change of variables $y = 2v, x = u$, i.e. $f(u, v) = u, 2v$. Then $\det[df] = 2$ and hence by the change of variables formula $\int_U y = \int_V 2 \cdot 2v = \int_V 4v$ where $V = \{(u, v) \in R^2|$ such that $0 < u^2 + v^2 < 1, y > 0, -v < u < v\}$. making another change of coordinates $u = r\cos\theta, v = r\sin\theta$ or $g(r, \theta) = (r\cos\theta, r\sin\theta)$ we see that $\det[dg] = r$ and $\int_V 4v = \int_W 4r \cdot r\sin\theta = \int_W 4r^2 \sin\theta$

where $W = \{0 < r < 1, \pi/4 < \theta < 3\pi/4\}$. By Fubini's theorem we compute

$$\int_W 4r^2 \sin\theta = \int_{\pi/4}^{3\pi/4} (\int_0^1 4r^2 \sin\theta dr) d\theta = \int_{\pi/4}^{3\pi/4} \frac{4}{3}\sin\theta d\theta = -\frac{4}{3}\cos\theta|_{\pi/4}^{3\pi/4} = \frac{4\sqrt{2}}{3}$$

(5) (15 pts) Let $v_1, \ldots v_{n-1}$ be vectors in R^n.

Show that $v_1 \times v_2 \times \ldots \times v_{n-1} = 0$ if $v_1, \ldots v_{n-1}$ are linearly dependent.

Solution

Let $v = v_1 \times v_2 \times \ldots \times v_{n-1}$. By definition,

$$\langle v, w\rangle = \det \begin{pmatrix} v_1 \\ \cdots \\ v_{n-1} \\ w \end{pmatrix} = 0$$

because the rows of this matrix are linearly dependent.

Thus $\langle v, w\rangle = 0$ for any $w \in R^n$. In particular for $w = v$ we get $0 = \langle v, v\rangle = |v|^2$ which means that $v = 0$.

(6) (15 pts) Let e_1, e_2 be a basis of a vector space V of dimension 2. Let $T \in \mathcal{T}^2(V)$ be given by $e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$.

Prove that T can not be written as $S \otimes U$ with $S, U \in \mathcal{T}^1(V)$.

Solution

Suppose $e_1^* \otimes e_1^* + e_2^* \otimes e_2^* = S \otimes U$ for some $S = ae_1^* + be_2^*$, $U = ce_1^* + de_2^*$. Then $S \otimes U = (ae_1^* + be_2^*) \otimes (ce_1^* + de_2^*) = ace_1^* \otimes e_1^* + bce_2^* \otimes e_1^* + ade_2^* \otimes e_1^* + bde_2^* \otimes e_2^* = e_1^* \otimes e_1^* + e_2^* \otimes e_2^*$. This means that $ac = 1, bc = 0, ad = 0, bd = 1$. It's easy to see that this system has no solutions. for example, $abcd = (bc)(ad) = 0 \cdot 0 = 0$ and on the other hand, $abcd = (ac)(bd) = 1 \cdot 1 = 1$. This is a contradiction.

(7) (15 pts) Let $\omega = e^{\frac{1}{\sqrt[3]{x^2+y^2}}}dx + \frac{\cos(\frac{\pi\sqrt{x^2+y^2}}{2})}{1+e^{x^2y}}dy$ be a 1-form on $R^2\backslash\{0\}$.

let $f\colon R^2 \to R^2\backslash\{0\}$ be given by $f(u,v) = (\cos(2u+v), \sin(2u+v))$.

Compute $f^*(d\omega)$.

Solution

First recall that $f^*(d\omega) = df^*(\omega)$. We compute $f^*(\omega) = e^1 d\cos(2u+v) + \frac{\cos(\pi/2)}{1+e^{\cos^2(2u+v)\sin(2u+v)}}d\sin(2u+v) = ed\sin(2u+v)$. therefore, $f^*(d\omega) = df^*(\omega) = d(ed\sin(2u+v)) = e(d\circ d)\sin(2u+v) = 0$