

MAT 257Y Solutions to Practice Final 2

- (1) Let $A \subset \mathbb{R}^n$ be a rectangle and let $f: A \rightarrow \mathbb{R}$ be bounded. Let P_1, P_2 be two partitions of A . Prove that $L(f, P_1) \leq U(f, P_2)$.

Solution

The statement is obvious if $P_1 = P_2$. In general, let P' be a common refinement of P_1 and P_2 . Then $L(f, P_1) \leq L(f, P')$. Indeed, for any rectangle Q' of P' contained in a rectangle Q of P_1 we have that $m(f, Q) \leq m(f, Q')$. therefore

$$\begin{aligned} L(f, P') &= \sum_{Q' \in P'} m(f, Q') \text{vol} Q' = \sum_{Q \in P} \sum_{Q' \subset Q} m(f, Q') \text{vol} Q' \geq \\ &\sum_{Q \in P} \sum_{Q' \subset Q} m(f, Q) \text{vol} Q' = \sum_{Q \in P} m(f, Q) \sum_{Q' \subset Q} \text{vol} Q' = \\ &\sum_{Q \in P} m(f, Q) \text{vol} Q = L(f, P_1) \end{aligned}$$

Thus $L(f, P_1) \leq L(f, P')$ and similarly $U(f, P') \leq U(f, P_2)$. This finally gives

$$L(f, P_1) \leq L(f, P') \leq U(f, P') \leq U(f, P_2)$$

- (2) Let $T: \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a 2-tensor on \mathbb{R}^n . Show that T is differentiable at $(0, 0)$ and compute $dT(0, 0)$.

Solution

Let $x = (x_1, \dots, x_n)$ be the coordinates on the first copy of \mathbb{R}^n and $y = (y_1, \dots, y_n)$ on the second. Then by multilinearity we have that $T(x, y) = \sum_{ij} T_{ij} x_i y_j$. This function is a polynomial and hence is differentiable. It is also obvious to check that its partial derivatives at $(0, 0)$ are all zero. therefore $dT(0, 0) = 0$.

- (3) Let $\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$ be a 2-form on $R^3 \setminus (0, 0, 0)$.

Verify that ω is closed.

Hint: One way to simplify the computation is to write $\omega = f \cdot \tilde{\omega}$ where $f = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$ and $\tilde{\omega} = xdy \wedge dz + ydz \wedge dx + zdx$.

Solution

We have $d\omega = df \wedge \tilde{\omega} + (-1)^0 f d\tilde{\omega}$

$$\begin{aligned} df \wedge \tilde{\omega} &= -\frac{3}{2} \frac{1}{(x^2 + y^2 + z^2)^{5/2}} (2xdx + 2ydy + 2zdz) \wedge (xdy \wedge dz + ydz \wedge dx + zdx) \\ &= -\frac{3(x^2 + y^2 + z^2)dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{5/2}} = -\frac{3dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}} \\ fd\tilde{\omega} &= \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \cdot 3dx \wedge dy \wedge dz = \frac{3dx \wedge dy \wedge dz}{(x^2 + y^2 + z^2)^{3/2}} \end{aligned}$$

Therefore $d\omega = df \wedge \tilde{\omega} + (-1)^0 f d\tilde{\omega} = 0$.

- (4) Let $f: R^2 \rightarrow R^2$ be given by $f(x, y) = (e^{2y}, 2x + y)$ and let $\omega = x^2 y dx + y dy$.

Compute $f^*(d\omega)$ and $d(f^*(\omega))$ and verify that they are equal.

Solution

$$\begin{aligned} d\omega &= (2xydx + x^2 dy) \wedge dx + dy \wedge dy = -x^2 dx \wedge dy. \\ f^*(d\omega) &= -(e^{2y})^2 de^{2y} \wedge (2dx + dy) = -2e^{6y} dy \wedge (2dx + dy) = 4e^{6y} dx \wedge dy. \end{aligned}$$

On the other hand, $f^*(\omega) = (e^{2y})^2(2x + y)de^{2y} + (2x + y)(2dx + dy) = 2e^{6y}(2x + y)dy + (2x + y)(2dx + dy)$.

Finally, $d(f^*(\omega)) = d(2e^{6y}(2x + y)) \wedge dy + (2dx + dy) \wedge (2dx + dy) = (12e^{6y}(2x + y)dy + 2e^{6y}(2dx + dy)) \wedge dy + 0 = 4e^{6y}dx \wedge dy$.

- (5) Determine if $\int_{0 < x^2 + y^2 < 1}^{ext} \ln(x^2 + y^2)$ exists and if it does compute it.

Solution

Let $U = 0 < x^2 + y^2 < 1 \setminus \{(0, 1) \times 0\}$.

then $\int_{0 < x^2 + y^2 < 1}^{ext} \ln(x^2 + y^2)$ exists iff $\int_U^{ext} \ln(x^2 + y^2)$ exists and if they both exist they are equal. For the second integral make a polar change of variables $x = r \cos \theta, y = r \sin \theta$ where $0 < r < 1, 0 < \theta < 2\pi$. Then $\int_U^{ext} \ln(x^2 + y^2) = \int_0^{2\pi} (\int_0^1 r \ln r^2 dr) d\theta = 4\pi \int_0^1 r \ln r dr = 4\pi \int_0^1 \ln r d(r^2/2) = 4\pi (\frac{r^2 \ln r}{2} \Big|_0^1 - \int_0^1 \frac{r^2}{2} d \ln r) = 4\pi (0 - \int_0^1 r/2 dr) = -4\pi r^3/6 \Big|_0^1 = -2\pi/3$. Here we used the fact that $\lim_{r \rightarrow 0+} r^2 \ln r = 0$.

Thus $\int_{0 < x^2 + y^2 < 1}^{ext} \ln(x^2 + y^2) = -2\pi/3$

- (6) Let U, V be open in R^n . Let $f: R^n \rightarrow R$ be a continuous nonnegative function such that $\int_U^{ext} f$ and $\int_V^{ext} f$ exist.

Prove that $\int_{U \cup V}^{ext} f$ exists.

Hint: use compact exhaustions of U and V to construct a compact exhaustion of $U \cup V$.

Solution

let K_i be a compact exhaustion by measurable sets of U and C_i be a compact exhaustion by measurable sets of V . then we have that $\int_{K_i} f$ is increasing and $\lim_{i \rightarrow \infty} \int_{K_i} f = \int_U^{ext} f$. Similarly, $\lim_{i \rightarrow \infty} \int_{C_i} f = \int_V^{ext} f$.

Then it's easy to see that $K_i \cup C_i$ is a compact exhaustion by measurable sets of $U \cup V$. Since $f \geq 0$ we have that $\int_{C_i \cup K_i} f = \int_{C_i} f + \int_{K_i} f - \int_{C_i \cap K_i} f \leq \int_{C_i} f + \int_{K_i} f \leq \int_U^{ext} f + \int_V^{ext} f$. Therefore $\lim_{i \rightarrow \infty} \int_{K_i \cup C_i} f$ exists and hence so does $\int_{U \cup V}^{ext} f$.

- (7) Let $F(x) = \int_{e^x}^{x^2} f(tx) dt$ where $f: R \rightarrow R$ is C^1 .

Show that $F(x)$ is C^1 and find the formula for $F'(x)$.

Solution

Let $G(x, a, b) = \int_a^b f(tx)dt$. Then G is C^1 by a theorem from class and. $\frac{\partial G}{\partial x}(x, a, b) = \int_a^b \frac{d}{dx}f(tx)dt = \int_a^b tf'(tx)dt$. Also, $\frac{\partial G}{\partial b}(x, a, b) = f(bx)$ and $\frac{\partial G}{\partial a}(x, a, b) = -f(ax)$.

Then $F(x) = G(x, e^x, x^2)$ is C^1 by the chain rule and $F'(x) = \frac{\partial G}{\partial x}(x, e^x, x^2) + \frac{\partial G}{\partial a}(x, e^x, x^2) \cdot (e^x)' + \frac{\partial G}{\partial b}(x, e^x, x^2) \cdot (x^2)'$
 $(x^2)' = \int_{e^x}^{x^2} tf'(tx)dt - f(e^x x)e^x + f(x^3)2x$.

- (8) Let $x(t_1, t_2) = t_1 \cos t_2$, $y(t_1, t_2) = t_1^2 + e^{t_1 t_2}$. Let $f(x, y)$ be a differentiable function $f: R^2 \rightarrow R$. Let $g(t_1, t_2) = f(x(t_1, t_2), y(t_1, t_2))$. Express $\frac{\partial g}{\partial t_1}(1, 0)$ and $\frac{\partial g}{\partial t_2}(1, 0)$ in terms of partial derivatives of f .

Solution

By the chain rule

$$\frac{\partial g}{\partial t_1}(t_1, t_2) = \frac{\partial f}{\partial x}(x(t), y(t)) \frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial y}(x(t), y(t)) \frac{\partial y}{\partial t_1} = \frac{\partial f}{\partial x}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2}) \cos t_2 + \frac{\partial f}{\partial y}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2}) (2t_1 + t_2 e^{t_1 t_2}).$$

$$\text{Similarly, } \frac{\partial g}{\partial t_2}(t_1, t_2) = \frac{\partial f}{\partial x}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2}) (-t_1 \sin t_2) + \frac{\partial f}{\partial y}(t_1 \cos t_2, t_1^2 + e^{t_1 t_2}) (t_1 e^{t_1 t_2}).$$

- (9) Mark true or false. Justify your answer.

Let A, B be any subsets of R^n .

- (a) $br(A) \subset Lim(A)$
- (b) $Lim(A) \subset A$
- (c) $br(A \cap B) \subset br(A) \cap br(B)$.

Solution

- (a) **False.** Example $A = \{p\}$. Then $br A = \{p\}$ and $Lim(A) = \emptyset$.
- (b) **False.** Example $A = (0, 1) \subset R$. Then $Lim(A) = [0, 1]$ is not contained in A .

- (c) **False.** Example $A = [0, 2], B = [1, 3]$. Then $A \cap B = [1, 2]$ and $br(A \cap B) = \{1, 2\}$. On the other hand, $br(A) \cap br(B) = \emptyset$.
- (10) let $M^2 \subset R^3$ be the torus of revolution obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the xz plane around the yz axis. Consider the orientation on M induced by the outward normal field N where $N(3, 0, 0) = (1, 0, 0)$.
Find $\int_M x dy \wedge dz$

Solution

Let V be the solid obtained by rotating the disk $U = (x - 2)^2 + z^2 \leq 1$ in the xz plane around the yz axis. then $M = \partial V$ and by Stokes' formula $\int_M x dy \wedge dz = \int_V d(x dy \wedge dz) = \int_V dx \wedge dy \wedge dz = \text{vol} V$. Recall that by a homework problem this is equal to $2\pi \int_U x$. Using polar coordinates change of variables $x = 2 + r \cos \theta, y = r \sin \theta$ we compute

$$\int_U x = \int_0^{2\pi} \int_0^1 (2 + r \cos \theta) r dr d\theta = \int_0^{2\pi} \int_0^1 (2r + r^2 \cos \theta) dr d\theta = 2\pi.$$

Therefore $\int_M x dy \wedge dz = 4\pi^2$.

- (11) Let $M \subset R^n$ be an oriented manifold.

Prove that $\text{vol}(M) = \int_M dA$ is positive.

Solution

Let U_i be a covering of M by orientation preserving coordinate patches $f_i: W_i \rightarrow M$ and let ϕ_i be the partition of unity subordinate to this covering. Note that $0 \leq \phi_i \leq 1$. Then $\int_M dA = \sum_i \int_M \phi_i dA = \sum_i \int_{W_i} f_i^*(\phi_i dA) = \sum_i \int_{W_i} (\phi_i \circ f_i) f_i^*(dA)$. Note that $\phi_i \circ f_i(x) \geq 0$ for any $x \in W_i$ and is positive at some point of W_i . We also have that $f_i^*(dA) = u(x) dx^1 \wedge \dots \wedge dx^k$ where $u(x) = dA(f_{i*}e_1, \dots, f_{i*}e_k) > 0$ since f_i is orientation preserving. Altogether the above means that $\int_{W_i} (\phi_i \circ f_i) f_i^*(dA) = \int_{W_i} g_i(x)$ where g_i is

a continuous nonnegative function with compact support which is positive somewhere. therefore $\int_{W_i} g_i(x) > 0$ and hence $\int_M dA = \sum_i \int_M \phi_i dA > 0$.