MAT 267F Solutions to the Practice Term Test

(1) Solve the following IVP:

$$\begin{cases} y' = 2\cos^2 y \cdot \sin(2x) \\ y(0) = 0 \end{cases}$$

What is the interval of existence of the solution?

Solution

By separating variables we get

$$\sec^2 y dy = 2\sin(2x)dx$$

which integrates to

$$\tan y = -\cos(2x) + C$$

From the initial conditions we get

$$\tan(0) = -1 + C, 0 = -1 + C, C = 1$$

which gives

$$\tan(y) = 1 - \cos(2x) + 1, \text{ or } y = \tan^{-1}(-\cos(2x) + 1)$$

It is obvious from this formula that the solution is defined for all real x.

(2) Using the variation of parameter find the general solution of the following equation:

$$y'' - y' - 2y = te^t$$

Solution

First we solve the homogeneous equation $y'' - y' - 2y = 0$. Its characteristic equation is $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$ so that the roots are $\lambda = 2, -1$ and hence the fundamental solutions are $y_1 = e^{2t}, y_2 = e^{-t}$. The Wronskian corresponding these solutions is equal to

$$W = \det \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} = -3e^t$$

By the variation of parameter method we can look for a solution of the original equation in the form $y = c_1(t)y_1(t) + c_2 y_2(t)$ where

$$c_1' = -\frac{y_2 te^t}{W} \quad c_2' = \frac{y_1 te^t}{W}$$

$$c_1' = -\frac{e^{-t}te^t}{-3e^t} = \frac{te^{-t}}{3}, c_1 = \int \frac{te^{-t}}{3}dt = -\frac{(t+1)e^{-t}}{3} + C_1$$

$$c_2' = \frac{e^{2t}te^t}{-3e^t}, c_2 = -\frac{1}{3}\int te^{2t}dt = -\frac{1}{6}te^{2t} + \frac{1}{12}e^{2t} + C_2$$

which finally gives

$$y = (-\frac{(t+1)e^{-t}}{3} + C_1)e^{2t} + (-\frac{1}{6}te^{2t} + \frac{1}{12}e^{2t} + C_2)e^{-t}$$

$$= -\frac{(t+1)e^t}{3} - \frac{1}{6}te^t + \frac{1}{12}e^t + C_1e^{2t} + C_2e^{-t}$$

(3) Mark true or false. If true give an explanation. If false, give a counterexample.

(a) For any real 2×2 matrix A and any nonzero solution of $y' = Ay$ we either have that $||y(t)|| \to 0$ or $||y(t)|| \to \infty$ as $t \to \infty$.

(b) Suppose $y(t)$ is a nonzero solution of $y' = Ay$ for some 2×2 matrix A such that $y(t) \underset{t\to\infty}{\longrightarrow} 0$. Then if we change the initial condition $y(0)$ slightly, the solution will still go to 0 as $t \to \infty$.
In other words, there exists $\epsilon > 0$ such that if $\tilde{y}' = A\tilde{y}$ and $||y(0) - \tilde{y}(0)|| < \epsilon$ then $\tilde{y}(t) \underset{t\to\infty}{\longrightarrow} 0$.

Solution

(a) **False**. If the phase portrait is a center, all nonzero solutions stay bounded but don't go to zero For example for the system

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

the integral curves of all non-constant solutions are circles centered at the origin.

(b) **False**. It fails for any saddle. For example, consider the system

$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

The solution with the initial condition $x(0) = 0, y(0) = 1$ is given by $x(t) = 0, y(t) = e^{-t}$ so that $(x(t), y(t)) \to (0,0)$ as $t \to \infty$.

However, if we vary the initial conditions slightly $x(0) = \epsilon, y(0) = 1$ then $x(t) = \epsilon e^{t}, y(t) = e^{-t}$ and $(x(t), y(t)) \nrightarrow (0,0)$ no matter how small ϵ is.

(4) Find the general solution of the following system

$$\begin{cases} y_1' = -y_1 + 5y_2 \\ y_2' = -2y_1 - 3y_2 \end{cases}$$

What type of phase portrait does this system have?

Solution

The matrix of this system is $\begin{pmatrix} -1 & 5 \\ -2 & -3 \end{pmatrix}$. We compute its eigenvalues and find that they are $\lambda = -2 \pm 3i$ and hence the phase portrait is a spiral sink. Since the vector filed at the point $(1,0)^t$ is equal to $(-1,-2)^t$ points down we see that it's a clockwise (i.e right) spiral.

To find the general solution we find the eigenvector corresponding to $\lambda = -2 + 3i$:

$$A - (-2 + 3i)I = \begin{pmatrix} 1 - 3i & 5 \\ -2 & -1 - 3i \end{pmatrix}$$

We find that $v = (5, 3i - 1)^t$ is an eigenvector. Hence $y(t) = e^{\lambda t}v$ is a solution of $y' = Ay$. The real and imaginary parts of $y(t)$ are also solutions. We compute

$$y(t) = e^{(-2+3i)t}\begin{pmatrix} 5 \\ 3i - 1 \end{pmatrix} = e^{-2t}(\cos 3t + i \sin 3t)\begin{pmatrix} 5 \\ 3i - 1 \end{pmatrix}$$

$$= e^{-2t}\begin{pmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + ie^{-2t}\begin{pmatrix} 5\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}$$

Hence the general solution is $c_1y_1(t) + c_2y_2(t)$ where

$$y_1(t) = e^{-2t}\begin{pmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix}, \quad y_2(t) = e^{-2t}\begin{pmatrix} 5\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}$$

(5) Find a 2×2 matrix A such that $y_1 = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ and $y_2 = \begin{pmatrix} e^{2t} \\ -2e^{2t} \end{pmatrix}$ satisfy the equation $y' = Ay$.

Solution

We can rewrite $y_1 = e^t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $y_2 = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Thus the matrix A should have eigenvectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with the corresponding eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$.

This means that $A = TDT^{-1}$ where $T = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Computing T^{-1} and multiplying the matrices we find

$$A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$$

(6) Find e^A for the following matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

We write A as $A = B + C$ with $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Since $BC = CB$ we have that $e^A = e^B e^C$. Clearly, $e^B = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^2 \end{pmatrix}$. Since $C^3 = 0$ we have that $e^C = I + C + \frac{C^2}{2} = \begin{pmatrix} 1 & 1 & \frac{5}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

Therefore,

$$e^A = \begin{pmatrix} e^2 & e^2 & \frac{5e^2}{2} \\ 0 & e^2 & -e^2 \\ 0 & 0 & e^2 \end{pmatrix}$$

(7) Let A be an $n \times n$ matrix. Prove that $\det(e^A) = e^{trA}$. (If you wish, you can assume that all the eigenvalues of A are distinct.)

Solution

If all the eigenvalues of A are distinct it is diagonolizable so that there exists T such that $A = TDT^{-1}$ and D is diagonal. Then $tr(T) = tr(D)$ and $e^A = Te^DT^{-1}$ so that $\det e^A = \det(T)\det(e^D)\det(T^{-1}) =$

det(e^D). Therefore, it's enough to show that $\det(e^D) = e^{trD}$. To see this observe that if the diagonal entries of D are $\lambda_1, \ldots, \lambda_n$ then e^D is diagonal with entries $e^{\lambda_1}, \ldots, e^{\lambda_n}$. Then $\det(e^D) = e^{\lambda_1} \cdot \ldots \cdot e^{\lambda_n} = e^{\lambda_1 + \ldots + \lambda_n} = e^{trD}$.

(8) Show that if $e^{tA}e^{tB} = e^{tB}e^{tA}$ for any real t then $AB = BA$.

Hint: Differentiate!!

Solution

Differentiating $e^{tA}e^{tB}$ once we find $(e^{tA}e^{tB})' = Ae^{tA}e^{tB} + e^{tA}Be^{tB}$. Differentiating it one more time we get

$(e^{tA}e^{tB})'' = (Ae^{tA}e^{tB} + e^{tA}Be^{tB})' = A^2e^{tA}e^{tB} + Ae^{tA}Be^{tB} + Ae^{tA}Be^{tB} + e^{tA}B^2e^{tB}$.

Evaluating at $t = 0$ we get that $(e^{tA}e^{tB})''(0) = A^2 + 2AB + B^2$. The same computation shows that $(e^{tB}e^{tA})''(0) = A^2 + 2BA + B^2$. Since $e^{tA}e^{tB} = e^{tB}e^{tA}$ we have that $(e^{tA}e^{tB})''(0) = (e^{tB}e^{tA})''(0)$ so that $A^2 + 2AB + B^2 = A^2 + 2BA + B^2$ and hence $AB = BA$.