Solutions to selected problems from homework 1

(1) Problem 1.1.6 from the book.

(i) Let us parameterize the ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ (where $p > q > 0$) as $\gamma(t) = (p\cos t, q\sin t)$. Let $\varepsilon = \sqrt{1 - \frac{q^2}{p^2}}$. And let $F_+ = (\varepsilon p, 0), F_- = (-\varepsilon p, 0)$ be the foci of the ellipse. Note that F_+, F_- lie inside the ellipse because $\varepsilon < 1$.
Let $v_\pm = \gamma(t) - F_\pm$ and let $d_\pm = |v_\pm|$ be the distance from $\gamma(t)$ to $F_\pm$. We will compute d_+ (the computation of d_- is identical. We have $v_+ = (p\cos t - \varepsilon p, q\sin t)$. Therefore,
$d_+ = \sqrt{(p\cos t - \varepsilon p)^2 + q^2\sin^2 t} = p\sqrt{(\cos t - \varepsilon)^2 + \frac{q^2}{p^2}\sin^2 t} =$
$p\sqrt{\cos^2 t - 2\varepsilon\cos t + \varepsilon^2 + (1-\varepsilon^2)\sin^2 t} = p\sqrt{\cos^2 t - 2\varepsilon\cos t + \varepsilon^2 + \sin^2 t - \varepsilon^2\sin^2 t} =$
$p\sqrt{1 - 2\varepsilon\cos t + \varepsilon^2(1-\sin^2 t)} = p\sqrt{1 - 2\varepsilon\cos t + \varepsilon^2\cos^2 t} = p\sqrt{(1-\varepsilon\cos t)^2} =$
$p(1 - \varepsilon\cos t)$.
Likewise, $d_- = p(1+\varepsilon\cos t)$ and hence $d_+ + d_- = p(1-\varepsilon\cos t) + p(1+\varepsilon\cos t) = 2p$ is independent of t.

(ii) We have that $\gamma'(t) = (-p\sin t, q\cos t)$. Observe that $N(t) = (q\cos t, p\sin t)$ is perpendicular to $\gamma'(t)$. Then $n(t) = \frac{N(t)}{|N(t)|}$ is a unit normal to $\gamma'(t)$. The distance $h_\pm$ from $F_\pm$ to the tangent line at t is equal to $h_\pm = |\langle |v_\pm, n(t)\rangle|$.
Therefore,

$$h_+ \cdot h_- = \frac{|\langle v_+, N(t)\rangle \cdot \langle v_-, N(t)\rangle|}{|N(t)|^2} \tag{1}$$

We first compute the denominator: $|N(t)|^2 = q^2\cos^2 t + p^2\sin^2 t = p^2(\frac{q^2}{p^2}\cos^2 t + \sin^2 t) = p^2((1-\varepsilon^2\cos^2 t + \sin^2 t) = p^2(1-\varepsilon^2\cos^2 t)$.
Next, let's compute the numerator
$|\langle v_+, N(t)\rangle \cdot \langle v_-, N(t)\rangle| = |\langle (p\cos t - \varepsilon p, q\sin t), (q\cos t, p\sin t)\rangle \cdot \langle (p\cos t + \varepsilon p, q\sin t), (q\cos t, p\sin t)\rangle| = |(pq(\cos t - \varepsilon)\cos t + pq\sin^2 t) \cdot (pq(\cos t + \varepsilon)\cos t + pq\sin^2 t)| = p^2q^2|(\cos^2 t - \varepsilon\cos t + \sin^2 t) \cdot (\cos^2 t + \varepsilon\cos t + \sin^2 t)| = p^2q^2(1-\varepsilon\cos t)(1+\varepsilon\cos t) = p^2q^2(1-\varepsilon^2\cos^2 t)$.
Plugging the above formulas into (1) we get

$$h_+ \cdot h_- = \frac{p^2q^2(1-\varepsilon^2\cos^2 t)}{p^2(1-\varepsilon^2\cos^2 t)} = q^2$$

which is independent of t.

(iii) We need to show that the two lines through the foci and $\gamma(t)$ make equal angles with the tangent line at $\gamma(t)$. This is equivalent to showing that $|\sin\angle v_+\gamma'(t)| = |\sin\angle v_-\gamma'(t)|$.

Recall that for any vectors u, v in $\mathbb{R}^2$ we have that $|u \times v| = |\det \begin{pmatrix} u \\ v \end{pmatrix}| = |u| \cdot |v| \cdot |\sin \angle uv|$.

Applying this to $u = \gamma'(t), v = v_+$ we get
$|\gamma'(t) \times v_+| = |(-p \sin t, q \cos t) \times (p \cos t - \varepsilon p, q \sin t)| = |-pq \sin^2 t - pq \cos t(\cos t - \varepsilon)| = pq(1 - \varepsilon \cos t)$.

Recall that by part i) we know that $|v_+| = d_+ = p(1 - \varepsilon \cos t)$. Therefore, $|\sin \angle \gamma'(t) v_+| = \frac{pq(1-\varepsilon \cos t)}{|\gamma'(t)| p(1-\varepsilon \cos t)} = \frac{q}{|\gamma'(t)|}$. An identical computation also shows that $|\sin \angle \gamma'(t) v_-| = \frac{q}{|\gamma'(t)|}$ which finally gives that
$|\sin \angle \gamma'(t) v_+| = |\sin \angle \gamma'(t) v_-|$.

(2) Let $\gamma \colon (\alpha, \beta) \to \mathbb{R}^2$ be a regular curve. Prove that for any $t_0 \in (\alpha, \beta)$ there is a small $\epsilon > 0$ such that the image $\gamma(t_0 - \epsilon, t_0 + \epsilon)$ can be written as the level set $f(x, y) = c$ where f is smooth and c is a regular value.

Solution

Let $\gamma(t) = (x(t), y(t))$. By the assumption $\gamma'(t_0) = (x'(t_0), y'(t_0)) \neq 0$.

Suppose $x'(t_0) \neq 0$ (the other case is handled similarly).

Then by the inverse function theorem $x = x(t)$ is invertible near t_0 and we can solve $t = t(x)$. Using this change of variable we can reparameterize γ near t_0 as $x \mapsto \gamma(t(x)) = (x, y(t(x)))$. In other words near t_0 our curve is the graph of $y = h(x)$ where $h = y(t(x))$.

Set $f(x, y) = y - h(x)$. Then the graph of h is equal to the level set $\{f = 0\}$ and 0 is clearly a regular value of f since $\nabla f = (-h'(x), 1) \neq 0$ for any $(x, y) \in \{f = 0\}$.