Countability of the Rational Numbers

by L. Shorser

Theorem: It is possible to count the positive rational numbers.

Proof. In order to show that the set of all positive rational numbers,

$$\mathbb{Q}_{>0} = \{\frac{r}{s} | r, s \in \mathbb{N}\}$$

is a countable set, we will arrange the rational numbers into a particular order. Then we can define a function f which will assign to each rational number a natural number. In other words, we will arrange the rationals in a way that allows us to count them.

We begin by creating a chart with the numerators ascending from left to right and denominators ascending from top to bottom:

$$\begin{array}{ccccc} 1/1 & 2/1 & 3/1 & 4/1 & \cdots \\ 1/2 & 2/2 & 3/2 & 4/2 & \cdots \\ 1/3 & 2/3 & 3/3 & 4/3 & \cdots \\ 1/4 & 2/4 & 3/4 & 4/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Now, we can see that it is possible to assign the number 1 to the top left corner. Then the number 2 to the 1/2 beneath 1/1, and 3 to the 2/1 above and to the right. We will continue by ordering each number along ascending diagonals starting in the first column and moving up and to the right, ignoring any fraction that has already been included on the list already. This effectively counts the positive rational numbers.

□

Structure:

Assumptions:

Definitions and Results:

Scope: