Induction and the Well-Ordering Principle

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According to www.mathworld.wolfram.com, the well-ordering principle says:

Every non-empty set of positive integers contains a smallest member.

It can also be stated for all sets, not just sets of integers and is related to Zorn's Lemma and the Axiom of Choice.

Goal of Induction: Induction is a proof technique that proves a statement for all values of n, where n is any natural number greater than or equal to a smallest number (the number guaranteed by the well-ordering principle).

Step 1: Prove the statement for the smallest value of n. This is called the *base case*.

Step 2: Assume that the statement is true when $n = k$. This is called the *induction hypothesis*.

Step 3: Use the induction hypothesis to prove that the statement is also true for $n = k + 1$.

In other words, we prove that the statement is true for the smallest value of n and that we can get from one value of n to the next (k to $k + 1$). This is the mathematical equivalent of proving you can climb a staircase by seeing if you can climb onto the first step and then measuring to make sure that the distance between steps is climbable.

Example: Prove that the sum of the numbers 1 to n is equal to $\frac{n(n+1)}{2}$.
Step 1) Base case: $1 = \frac{1(1+1)}{2}$
Step 2) Induction Hypothesis: $1 + \ldots + k = \frac{k(k+1)}{2}$
Step 3) Case where $n = k + 1$:
Left Side $= 1 + \ldots + k + (k + 1)$
$= \frac{k(k+1)}{2} + (k + 1)$
$= \frac{1}{2}(k^2 + k) + \frac{1}{2}(2k + 2)$
$= \frac{1}{2}(k + 1)(k + 2)$
$= \frac{(k+1)((k+1)+1)}{2}$ Q.E.D