COMPLEX *A PRIORI* BOUNDS REVISITED

MICHAEL YAMPOLSKY

RÉSUMÉ. Nous revisitons l'existence de bornes complexes pour la renormalisation des polynômes quadratiques réels. Nous donnons une nouvelle preuve de notre résultat avec Lyubich pour les polynômes quadratiques de type essentiellement borné à l'aide d'un argument fondé sur l'étude de la géométrie des ensembles de Julia paraboliques.

ABSTRACT. We revisit the question of the existence of complex *a priori* bounds for renormalizations of real quadratic polynomials. We give a new proof of our joint result with Lyubich for the quadratics of essentially bounded type with an argument based on the study of the geometry of parabolic Julia sets.

1. INTRODUCTION

This paper addresses the well-studied problem of the existence of complex *a priori* bounds in the dynamics of quadratic polynomials. By definition, an infinitely renormalizable quadratic map f has such bounds if there exists a lower bound $\mu > 0$ such that for every $n \in \mathbb{N}$ the renormalization $\mathcal{R}^n f$ has a quadratic-like extension $U \to V$ whose fundamental annulus $V \setminus U$ has modulus at least μ. The purpose of establishing such bounds is two-fold: they were originally introduced by Sullivan [Sul1, Sul2, MvS] as a compactness condition for the one-dimensional renormalization theory; on the other hand the geometric control they give leads to rigidity results, such as JLC, and MLC (see e.g. [Lyu4]). The problem of existence of complex *a priori* bounds for real infinitely renormalizable quadratics was completely settled following Sullivan's original result for quadratics of bounded type [Sul2, MvS], in the works [Lyu3, LvS, LY, GS]:

Theorem 1.1. *There exists $\mu > 0$ such that for every infinitely renormalizable real quadratic polynomial f and every $n \in \mathbb{N}$ the renormalization $\mathcal{R}^n f$ has a quadratic-like extension with modulus at least μ.*

In §2 we discuss the history of the proof in some detail, and, in particular, introduce the combinatorial condition of *essentially bounded type*, which was the subject of study in [LY]. In this paper we give a new treatment to polynomials satisfying this condition. Our approach is to consider them as small perturbations of parabolic maps, and use the rigidity properties of such maps to pass from real *a priori* bounds to complex ones. A particularly simple proof of complex bounds for parabolic maps is due to Petersen in the case of critical circle maps (see [EY]). Slightly more work has to be done to get bounds

Date: November 17, 2003.
The author gratefully acknowledges the support of NSERC and the Connaught Foundation.

for quadratics (partly because the combinatorics is more complex) – however, the resulting argument is "soft", as opposed to a "hard" analytic proof given in [LY]. We note, that our proof accomplishes less than that of [LY], yet enough to replace the result of that paper.

Having such a geometric proof is interesting in itself, and draws an instructive parallel with the critical circle maps case. The study of geometric limits of renormalizations of quadratic-like maps with essentially bounded type was carried out by Hinkle [Hin], based on the *a priori* bounds of [LY]. Such limits are represented by towers of quadratic-like maps, similar to McMullen towers [McM2], but with parabolic elements. It is worth noting, that using our argument, we can replace the study of these towers by the analysis of the appropriate bi-infinite Epstein towers [Ep], similarly to the way the analysis in [EY] replaces [Ya]. It is also our hope that this approach will prove useful in other situations where the existence of complex *a priori* bounds is not yet known: such, for example, as non-real quadratics whose renormalizations are small perturbations of parabolics.

Acknowledgement. I would like to thank Xavier Buff for his valuable comments on the paper, which helped to greatly streamline the argument.

2. Preliminaries

2.1. **Generalities.** The knowledge of the theory of parabolic bifurcation in one dimension will be assumed throughout this paper. As a general reference, we recommend the paper [Sh]; all the relevant facts may be found there. In addition, a detailed study of the properties of Écalle-Voronin maps was carried out in the dissertation [Ep], which may also be of interest to a reader of this work. We will also assume that the reader is familiar with the subject of renormalization of unimodal and quadratic-like maps. We will generally follow the notation of [LY, Lyu6, Hin]. In particular, we will denote $\mathcal{E}$ the Epstein class, and $\mathcal{E}_s$ an Epstein class with a geometric bound s; $\mathcal{R}f$ the renormalization of a renormalizable unimodal map f, and $p\mathcal{R}f$ its pre-renormalization, that is, the non-rescaled first return map. A parabolic renormalization of a quadratic-like map in $\mathcal{E}$ will be denoted $\mathcal{P}_\theta f$, $\theta \in \mathbb{T}$, and $p\mathcal{P}_\theta f$ will again stand for the pre-renormalization. As usual, $\mathbb{C}_J$ will denote the complex plane with two slits on the sides of the interval J:

$$\mathbb{C}_J = (\mathbb{C} \setminus \mathbb{R}) \cup J.$$

A map $f \in \mathcal{E}$ is a double covering of a domain $\Omega_F \subset \mathbb{C}_I$ over $\mathbb{C}_J$, where $I \Subset J$, branched at 0. The combinatorial type of a renormalizable unimodal map f will be denoted $\tau(f)$; $\chi(f)$ will denote the straightening of a quadratic-like map f with a connected Julia set. We assume the real *a priori* bounds; the reader can find the proof in e.g. [MvS].

2.2. Essentially bounded combinatorics.

Definition of the essential period. A detailed discussion of the combinatorics of the puzzle of a unimodal map goes beyond the scope of this paper. We will assume that the reader is broadly familiar with the subject and will recall only briefly the main concepts as we encounter them. For a more detailed introduction we particularly recommend to the reader the recent paper of Lyubich [Lyu6]. In this chapter we will briefly recall the definition of the essential period of a renormalizable unimodal map, and discuss an example of an infinitely renormalizable unimodal map with essentially bounded combinatorics. We will follow the above mentioned work of Lyubich, and a detailed paper of Hinkle [Hin].

Let f be a renormalizable unimodal map, which for simplicity will be assumed to be even. The *principal nest* of f is the sequence of intervals

$$[\alpha(f), -\alpha(f)] \equiv I^0 \supset I^1 \supset I^2 \supset \cdots$$

where $\alpha(f)$ is the dividing fixed point of f, and $I^m \ni 0$ is the central component of the first return map of I^{m-1},

$$g_m : \cup I_i^m \to I^{m-1}.$$

A level $m > 0$ is *non-central*, if $g_m(0) \in I^{m-1} \setminus I^m$. If m is non-central, then $g_{m+1}|_{I^{m+1}}$ is not merely a restriction of the central branch of g_m, but a different iterate of f. Set $m(0) = 0$, and let

$$m(0) < m(1) < m(2) < \cdots < m(\kappa)$$

be the sequence of non-central levels. The map

$$g_{m(\kappa)+1}|_{I^{m(\kappa)+1}} \equiv p\mathcal{R}f.$$

For $0 \leq k < \kappa$ the nested intervals

$$I^{m(k)+1} \supset I^{m(k)+2} \supset \cdots \supset I^{m(k+1)}$$

form a *central cascade*, whose *length* is $m(k+1) - m(k)$. Lyubich called a cascade *saddle-node* if $0 \notin g_{m(k)+1}(I^{m(k)+1})$, otherwise he called it *Ulam-von Neumann*. The reason for this terminology is that if the length of a saddle-node cascade is large, then $g_{m(k)+1}|_{I^{m(k)+1}}$ is combinatorially close to the saddle-node quadratic map $x \mapsto x^2 + 1/4$; in the Ulam-von Neumann case the map is close to the Ulam-von Neumann map $x \mapsto x^2 - 2$.

Let $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$ and set $d_k(x) = \min\{j - m(k), m(k+1) - j\}$, where $g_{m(k)+1}(x) \in I^j \setminus I^{j+1}$. This number shows how deep the image of x lands inside the cascade. Let us now define d_k as the maximum of $d_k(x)$ over all points $x \in P(f) \cap (I^{m(k)} \setminus I^{m(k)+1})$. For a saddle-node cascade the levels l such that $m(k) + d_k < l < m(k+1) - d_k$ are *neglectable*. Now we define the essential period of f as follows. Set $J = I^{m(\kappa)+1}$, and let p be its period, that is the smallest positive integer for which $f^p(J) \ni 0$. Consider the orbit $J_0 \equiv J$, $J_i = f^i(J_0)$, $i \leq p-1$. Suppose that J_k lands at a neglectable level of a central cascade generated by the branch of $g_m|_{I^m} \equiv f^{l_m}$. In that case we will call the iterates J_k, $J_{k+1}, \ldots, J_{k+l_m-1}$, which constitute one iterate by the cascade, neglectable. The number of non-neglectable intervals in the orbit $\{J_i\}_{i=0}^{p-1}$ is the *essential period*, $p_e(f)$. Recall that an infinitely renormalizable map f has a bounded combinatorial type if there is a finite upper bound on the periods of its renormalizations. Similarly, f is said to have an *essentially bounded combinatorial type* if $\sup_k p_e(\mathcal{R}^k f) < \infty$.

An example of a map with essentially bounded combinatorics. The definiton given above is rather delicate. It is useful therefore to provide the reader with a simple yet archetypical example of an infinitely renormalizable map of unbounded but essentially bounded combinatorial type (cf. [Hin]). This map is constructed in such a way that its every renormalization is a small perturbation of a unimodal map with a period 3 parabolic orbit. Closeness to a parabolic will ensure that the renormalization periods are high, but the essential periods will all be bounded.

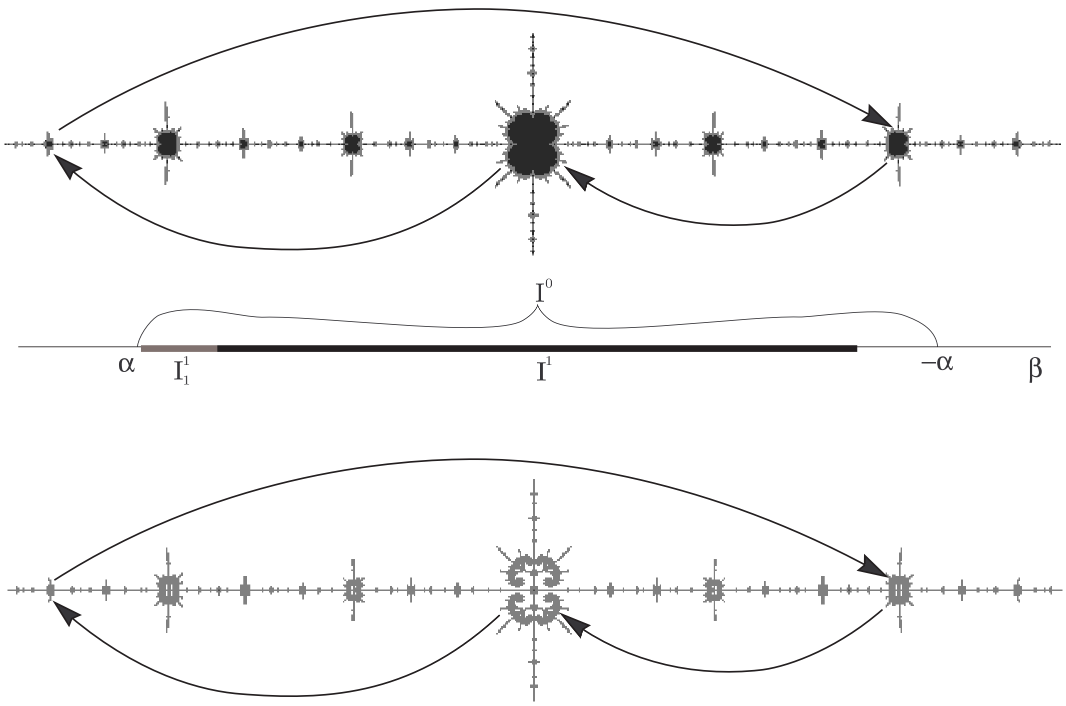

FIGURE 1. Construction of an example: The map $z \mapsto z^2 - 1.75$, and its small perturbation, with the domain of g indicated.

Before constructing the example, let us consider the dynamics of the quadratic map $f : z \mapsto z^2 - 1.75$. This polynomial has a parabolic orbit of period 3 on the real line, let us denote p the element of this orbit which is nearest to 0. Recall that $I^0 = [\alpha(f), -\alpha(f)]$, and I^1 is the central component of the domain of the first return map $g : I^0 \to I^0$. For this map we have $g|_{I^1} \equiv f^3$, $p \in I^0$, and $f^{3n}(0) \to p$. The map g has two non-central components; denoting I_1^1 the one whose boundary contains $\alpha(f)$, we have $g = f^2 : I_1^1 \to I^0$. For a small $\epsilon > 0$ let us set $f_\epsilon(z) = z^2 - 1.75 + \epsilon$. The orbit of 0 under f_ϵ eventually escapes I^0. Let us define ϵ_n as the parameter value for which $f_{\epsilon_n}^{3i}(0) \in I^1$, $i \leq n-1$, $f_{\epsilon_n}^{3n}(0) \in I_1^1$, and $f_{\epsilon_n}^{3n+2}(0) = 0$. These maps correspond to the centers of a sequence of small copies $\mathcal{M}_n^{(3)}$ of the Mandelbrot set converging to he cusp $c = -1.75$ of the real period 3 copy $\mathcal{M}^{(3)}$.

For each f_{ϵ_n}, and every $x \in I^1 \setminus I^2$, the depth $d_1(x) = 0$. Therefore, all of the levels of the central cascade of f_{ϵ_n} are neglectable. Denote $J \equiv I^{m(2)}$ the renormalization interval of f_{ϵ_n}; its period is $p = p(f_{\epsilon_n})$, which is some large number if ϵ_n is small. To calculate the essential period of the orbit $\{J_i \equiv f^i_{\epsilon_n}(J)\}_{i=0}^{p-1}$, we have to ignore all of the iterates, but five: J_0, J_1, J_2, and J_{p-2}, J_{p-1}. These comprise the first iterate of the orbit of J by the cascade generated by the central branch of $g|_{I^1} \equiv f^3$, and the one iterate by the non-central branch $g|_{I^1_1} \equiv f^2$, which, after the interval has run through the whole cascade, maps it back over the critical point. Thus, the essential period $p_e(f_{\epsilon_n}) = 5$, on the other hand, obviously, $p(f_{\epsilon_n}) \to \infty$.

Now consider an infinitely renormalizable unimodal map h such that the combinatorial type $\tau(\mathcal{R}^k h) = \tau(f_{\epsilon_{n_k}})$, with $n_k \to \infty$. This is the desired example. We can, of course, select h in the real quadratic family, picking an infinitely renormalizable parameter value $c \in \mathcal{M}$ such that $\chi(\mathcal{R}^k(f_c)) \in \mathcal{M}^{(3)}_{n_k}$. This amounts to blowing up a small copy $\mathcal{M}^3_{n_1}$, finding its period 3 cusp, and the corresponding sequence of small copies converging to this cusp, blowing up one of them, *ad infinitum.*

2.3. **Complex** ***a priori*** **bounds.** By real *a priori* bounds, there exists $\sigma > 0$ such that the renormalizations of any infinitely renormalizable map in $\mathcal{E}$ are eventually in $\mathcal{E}_\sigma$. Complex *a priori* bounds were introduced by Sullivan, who (in collaboration with de Melo) proved the following theorem:

Theorem 2.1 ([Sul2, MvS]). *For every $p \in \mathbb{N}$ there exists $N = N(p) \in \mathbb{N}$, and $\mu = \mu(p) > 0$, such that if $f \in \mathcal{E}_\sigma$ is at least N times renormalizable, and*

$$p(\mathcal{R}^i f) < p \text{ for } i = 0, \ldots, N-1, \text{ then } \operatorname{mod} \mathcal{R}^N(f) > \mu.$$

Subsequently, Lyubich has shown:

Theorem 2.2. *There exists $p_0 \in \mathbb{N}$, $\mu_0 > 0$ such that if $f \in \mathcal{E}_\sigma$ is renormalizable, and if*

$$p_e(f) > p_0, \text{ then } \operatorname{mod}(\mathcal{R} f) > \mu_0.$$

The gap between the two theorems was filled in [LY] where a universal complex *a priori* bound was obtained for maps with essential periods bounded by p_0. In particular, [LY] contained a simple proof of Theorem 2.1 with a universal constant μ. Independently, different proofs of universal *a priori* bounds were given by Graczyk & Swiatek [GS], and Levin & van Strien [LvS]. In this paper we again look at the old problem of the gap between Theorem 2.1 and Theorem 2.2, and give a different argument for bridging the gap. Our argument is less general than that of [LY], since it requires that while the essential periods of the renormalized maps are bounded, the periods of renormalizations are sufficiently high, so the renormalizations are uniformly close to parabolics.

Let us define $\mathcal{L} \subset \mathcal{E}_\sigma$ to be the set of all limit points of infinitely renormalizable quadratic-like maps. The theorem we prove is the following:

Theorem 2.3. *For every $\kappa \in \mathbb{N}$, $\kappa \geq 3$, there exists $\mu = \mu(\kappa) > 0$ such that the following holds. Denote $\mathcal{L}_\kappa$ the set of maps $g \in \mathcal{L}$ with the property, that there exists a sequence $\{f_i\}_{i=-\infty}^0 \subset \mathcal{L}$, with $f_0 = g$ such that every f_i has a parabolic periodic orbit of period*

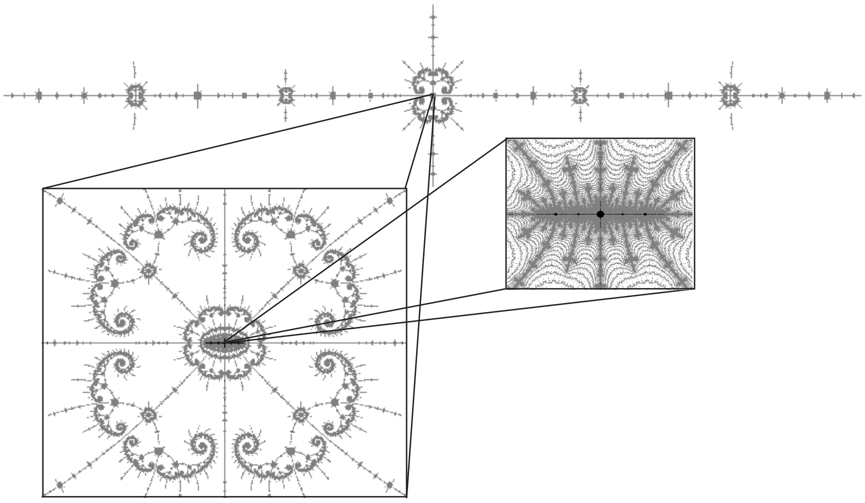

FIGURE 2. An airplane inside of an airplane: consecutive blow-ups of a Julia set of a map with essentially bounded combinatorics, and the corresponding blow-ups of the Mandelbrot set

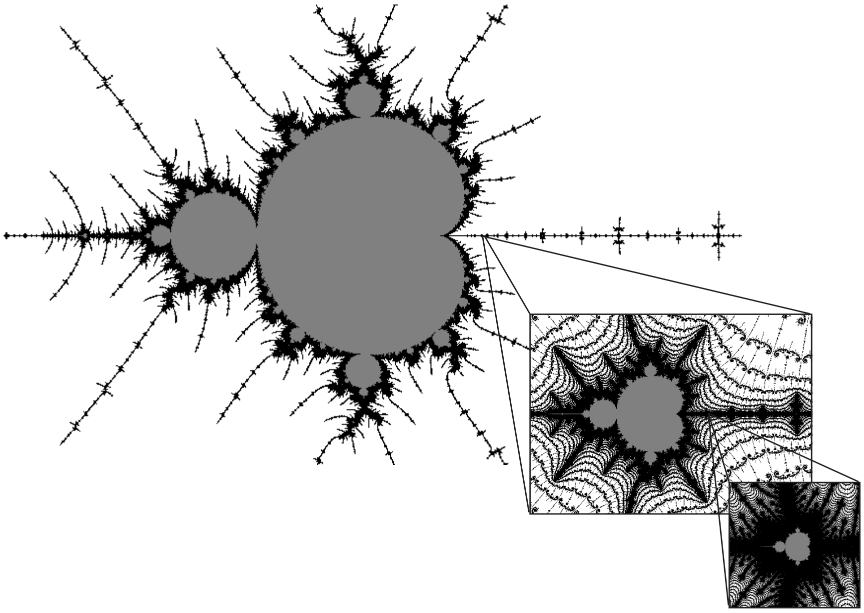

FIGURE 3

at most κ, and that for every i there exists θ_i such that the parabolic renormalization $\mathcal{P}_{\theta_i}(f_i) = f_{i+1}$.
Then $\operatorname{mod} g > \mu$ *for every* $g \in \mathcal{L}_\kappa$.

Note that the parabolic cycle of f_i is necessarily unique (cf. the argument in [Ya], as well as Lemma 3.2).

3. The proof of bounds

Outline of the argument. Since $\mathcal{L} \subset \mathcal{E}_\sigma$, it follows, in particular, that every map $f \in \mathcal{L}$ is an analytic double covering, branched at the origin, of a domain $\Omega = \Omega_f \subset \mathbb{C}_{I_f}$ over $\mathbb{C}_{J_f}$ with $I_f \Subset J_f$.

Let us fix $\kappa \in \mathbb{N}$, as in Theorem 2.3. For a map $f \in \mathcal{L}_\kappa$ denote $p = p(f)$ the period of its parabolic orbit. Let $B_f \subset \Omega_f$ be the parabolic basin of f, and B_f° the component of the immediate basin which contains the origin. We let $x_0 \in \partial B_f^\circ$ be the element of the parabolic orbit of f contained in the central component of the basin. Since $f : \Omega_f \to \mathbb{C}_{J_f}$ is a branched covering, $f : B_f^\circ \to B_f^\circ$ is a proper map in $\mathbb{C}_{J_f}$ compactified by adding the banks of the slits and the point at infinity. Further, let U_f^A, U_f^R be a pair of attracting and repelling petals of the parabolic point x_0; $\Phi_{A,f} : U_f^A \to \mathbb{C}$, $\Phi_{R,f} : U_f^R \to \mathbb{C}$ the corresponding Fatou coordinates; and $C_f^A \simeq \mathbb{C}/\mathbb{Z}$, $C_f^R \simeq \mathbb{C}/\mathbb{Z}$ the two Fatou cylinders. For each of the cylinders let $\oplus$, $\ominus$ denote their ends, correspondingly, the upper and the lower ones. The natural projection

$$\pi_{A,f} \equiv \Phi_{A,f} \operatorname{mod} \mathbb{Z} : U_f^A \to C_f^A$$

dynamically extends to a branched covering map of the whole basin B_f over C_f^A, the other projection, $\pi_{R,f}$ is only well-defined locally. Let $\mathcal{E}_f : C_f^R \to C_f^A$ be the dynamical first entry map, which we will further refer to as the *Écalle-Voronin map of f*. For ease of reference, let us summarize some of the relevant properties of $\mathcal{E}_f$ as a proposition.

Proposition 3.1 (Properties of Écalle-Voronin maps). *Under the above assumptions on f we have the following:*

(I) *the interior of the domain of the map $\mathcal{E}_f$ consists of the union of two open neighborhoods $U(\oplus)$, $U(\ominus)$ of the ends of the cylinder (two "polar caps"); and a countable set of topological disks $W_i \subset C^R$, each of which is a projection $\pi_{R,f}$ of a connected component B_i of B_f, intersecting U^R;*

(II) *the map $\mathcal{E}_f$ restricted to each of the interior components of its domain of definition is an infinite degree branched covering with a single critical value $v \in C^A$ (the projection $\pi_{A,f}(f^p(0))$), and infinitely many simple critical points.*

Note that the restrictions of $\mathcal{E}_f$ to the two polar caps are the original Écalle-Voronin conjugacy invariants, hence our choice of name for $\mathcal{E}_f$.

Now let us fix $f = f_0 \in \mathcal{L}_\kappa$, and let $f_{-1}, f_{-2}, \ldots$ be its preimages under the parabolic renormalization as in Theorem 2.3. Denote $W \ni 0$ the central component of the domain of

$\mathcal{E}_{f_{-1}}$. If we set $p\mathcal{P}_{\theta_{-1}}(f_{-1}) = \hat{f}$ (so f is a linear rescaling of $\hat{f}$), then the map $\hat{f}$ is conjugate via the projection $\pi_{A,f_{-1}}$ to the composition

$$h_f \equiv \mathcal{E}_{f_{-1}} \circ \tau_{\theta_{-1}} : \tilde{W} \equiv \tau_{\theta_{-1}}^{-1}(W) \to C^A_{f_{-1}}. \tag{3.1}$$

By Proposition 3.1 (II), the map h_f is an infinite degree branched covering with a single critical value. We will demonstrate that this map has a quadratic-like restriction with a definite modulus. To that end, we will first employ the real *a priori* bounds to show that the shape of the basin $B^\circ_{f_{-1}}$ (and hence the domain $W = \pi_{R,f_{-1}}(B^\circ_{f_{-1}})$) is geometrically bounded, and enclose it with an annulus of a definite modulus. We will then find a conformal preimage of this annulus inside a fundamental annulus for h_f.

The restriction on the period of parabolics in $\mathcal{L}_\kappa$ implies that h belongs to one of finitely many topological classes. We are actually able to show that it belongs to one of finitely many K-quasiconformal classes with a certain universal constant $K > 1$ (which is, obviously, a stronger statement than the existence of a quadratic-like restriction with a definite modulus). To do this, we apply a modified pull-back argument, along the lines of [EY], to quasiconformally conjugate our map to a fixed Écalle-Voronin map.

Bounding the shape of the parabolic basin of f_{-1}.

Lemma 3.2. *There exists $m = m(\kappa) > 0$ such that the following holds. Let $g \in \mathcal{L}_\kappa$, then*

$$m_g \equiv \operatorname{mod} \pi_{R,g}(U^R_g \setminus B^\circ_g) > m.$$

Proof. By compactness of $\mathcal{E}_\sigma$ it suffices to show that m_g is always a positive number. We will argue by way of contradiction. If $m_g = 0$, then the boundary of the basin component B°_g contains a point $z_0 \in U^R_g \cap \mathbb{R}$. Recall that g^p is the iterate fixing B°_g. Denote $I \subset [-\beta_g, \beta_g]$ the maximal interval of the unimodal branch of g^p, fixing B°_g. Given the invariance of ∂B°_g, the points $z_n = g^{pn}(z_0) \subset I$ converge to a point $\zeta \in \mathbb{R}$ which is fixed under the iterate g^p. Since g is a limit of a sequence of infinitely renormalizable quadratics without any attracting fixed points, ζ is necessarily parabolic. Since $g \in \mathcal{E}$, the iterate g^p has a univalent inverse branch in the upper half plane $\psi : \mathbb{H} \to \mathbb{H}$, fixing x_0. By real symmetry, there are points in $\mathbb{H}$ whose orbits under ψ converge to ζ. On the other hand, the point x_0 attracts some of the orbits in B°_g. This contradicts the uniqueness part of the Denjoy-Wolff Theorem as applied to ψ. □

Lemma 3.3. *There exists a constant $C = C(\kappa) > 0$ such that for every $g \in \mathcal{L}_\kappa$*

$$\operatorname{diam}(B^\circ_g) < C.$$

Proof. Again, compactness of $\mathcal{E}_\sigma$ means that it is enough to show $\operatorname{diam}(B^\circ_g) < \infty$. We argue by contradiction using the fact that g is a parabolic renormalization $g = \mathcal{P}_\theta g_{-1}$ of a map $g_{-1} \in \mathcal{L}_\kappa$. Let $h_g : \tilde{W} \to \mathbb{C}/\mathbb{Z}$, and $\hat{g}$ be as above (3.1). This map itself has a parabolic orbit in $\mathbb{R}/\mathbb{Z}$ (which is the projection of the parabolic orbit of g). Denoting $B \ni 0$ the central component of the parabolic basin of h_g we have

$$\pi_{A,g_{-1}}(B^\circ_{\hat{g}}) = B.$$

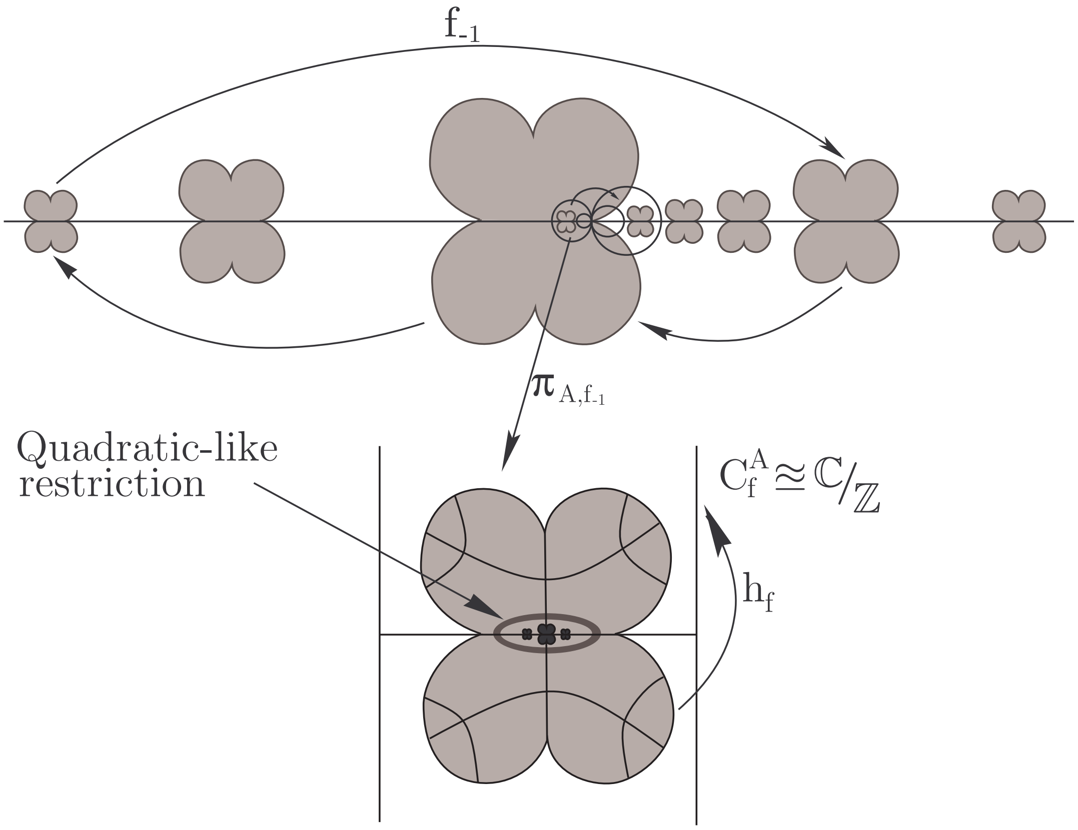

FIGURE 4. A sketch of the Julia set of f_{-1} and the domain of the map h_f. A quadratic-like restriction of h_f and its Julia set are also indicated.

Let us first observe that

$$\operatorname{diam}(B_g^\circ) = \infty$$

implies that ∂B, and hence ∂W as well, separates the cylinder. In view of the Maximum Modulus Principle the latter is equivalent to the existence of an equatorial continuum $X \subset \partial B \cap \partial W$.

We argue by way of contradiction. Assuming that there is no such equatorial continuum we observe that there exists a vertical strip

$$S_{[-N-k,-N]} = \{z \in \mathbb{C} | \; -N-k < \operatorname{Re}(z) < -N\},$$

such that there is a lift $\hat{B}$ of the basin B entirely contained in $(\pi_{A,g_{-1}})^{-1}(S_{[-N-k,-N]})$. On the other hand, since the two polar caps do not intersect W, the height of B is bounded, and hence

$$\hat{B} \subset (\pi_{A,g_{-1}})^{-1}(S_{[-N-k,-N]} \cap \{|\operatorname{Im}(z)| < M\}),$$

for some $M > 0$. Therefore, the conformal map $\hat{B} \to B_g^\circ$ can be extended to an open neighborhood, and hence the latter set has a finite diameter.

Let us now rule out the existence of a separating continuum X as above. Indeed, it would imply that there exists a bounded component S of $\mathbb{C}/\mathbb{Z} \setminus \operatorname{cl} B$ whose boundary contains the first preimage of the parabolic point $x \in \partial B$ under $h_g^p(g)$. The invariance of ∂B under $h_g^p(g)$ implies that the image $h_g^p(g)(X)$ intersects with $\mathbb{R}/\mathbb{Z}$ in the repelling petal of B. This, of course, means that $m_g = 0$, in contradiction with the previous lemma.

□

Projecting to the Fatou cylinders, we have an immediate corollary:

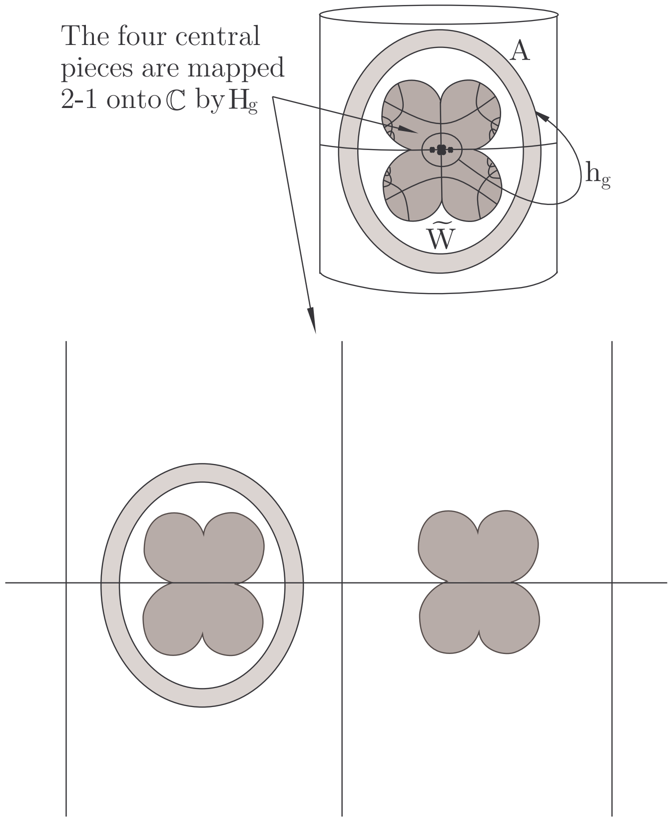

FIGURE 5

Corollary 3.4. *There exists $\mu = \mu(\kappa) > 0$ such that for every $g \in \mathcal{L}_\kappa$ the following holds. Let $\tilde{W}$ be the domain of h_g as in (3.1). Then there exists an annulus $A \subset C_{f^A_{-1}}$ enclosing*

$\tilde{W}$ *and such that*

$$\operatorname{mod} A > \mu.$$

We can now prove our main theorem:

Proof of Theorem 2.3. Let A be as above, and denote $V \ni 0$ the domain enclosed by the outer boundary of A. There exists a domain $\tilde{W} \supset U \ni 0$ which is a the two-fold preimage of V under h_g. To visualize this, lift the map h_g to $H_g : \tilde{W} \to \mathbb{C}$. Then the preimages of $\mathbb{R}$ in $\tilde{W}$ partition this domain into an infinite union of topological disks each of which is mapped onto the upper or the lower half-plane (see Figure 5). By construction, $U \subset \tilde{W}$, and hence $\operatorname{mod} V \setminus U > \mu$. □

Quasiconformal conjugacies.
Below we will sketch the proof of the following stronger result:

Theorem 3.5. *There exists $K = K(\kappa) > 1$ such that for every $g \in \mathcal{L}_\kappa$ the branched covering h_g belongs to one of finitely many K-quasiconformal types.*

Lemma 3.6. *There exists $K = K(\kappa) > 1$ such that the following holds. Let $g \in \mathcal{L}_\kappa$ and let $p = p(g) \in \mathbb{N}$ as before denote the period of B°_g. Then there exists a K-quasiconformal map of the plane which maps the basin*

$$B^\circ_g \to K(z^2 + 1/4),$$

conjugating the dynamics of g^p and $z^2 + 1/4$ on the respective basins.

Proof. The previous two Lemmas allow us to construct a pinched quadratic-like restriction of g^p on a neighborhood B°_g with universal quasiconformal bounds. We refer the reader to [EY] where the relevant definition is given and a similar construction is carried out. The pull-back argument of [EY] applied to the two pinched quadratic-like maps applies here *mutatis mutandis.* □

Proof of Theorem 3.5. Let f, f_{-1} be as above. Let Y_κ denote the set of real quadratic polynomials with a parabolic cycle of period at most κ; of course, $\#Y_\kappa < \infty$. Let $g_{-1} \in Y_\kappa$ be the map having the same combinatorial type as f_{-1}, and let $g = \mathcal{P}_\theta g_{-1}$ be the parabolic renormalization of g_{-1} having the same combinatorial type as f. Then h_g, h_f are K_1-quasiconformally conjugate with K_1 depending on κ alone. Indeed, this follows from the previous lemma, and the standard pull-back argument applied to h_g, h_f. □

Conclusion. Let us show that the existence of universal complex *a priori* bounds stated in Theorem 1.1 follows from our Theorem 2.3 together with Sullivan's Theorem 2.1 and Lyubich's Theorem 2.2. Indeed, let f be an infinitely renormalizable real quadratic map. If $p_e(\mathcal{R}^n f) > p_0$ then

$$\operatorname{mod}(\mathcal{R}^n f) > \mu_0$$

by the result of Lyubich. By the Sullivan's theorem, if f has sufficiently many consecutive renormalizations with a bound on the period, the last of them has a universal bound on the modulus. On the other hand, by real *a priori* bounds, all renormalizations of f with large periods and bounded essential periods are uniformly small perturbations of parabolics. Putting this together, we see that there exist natural numbers p_1, κ, N such that the following is true. If $p(\mathcal{R}^{n+i}f) < p_1$ for $1 \leq i \leq N-1$, then the conditions of Sullivan's theorem hold for $\mathcal{R}^{n+N}f$. If there exists $n < n+i \leq n+N-1$ such that both for $\mathcal{R}^n f$ and $\mathcal{R}^{n+i}f$ the periods $p > p_1$, but the essential periods $p_e < p_0$; and for all $0 < j < i$ the periods $p(\mathcal{R}^{n+j}f) < p_1$, then $\mathcal{R}^{n+i+1}f$ is a small perturbation of a map $g \in \mathcal{L}_\kappa$. Moreover, if we denote U and V the domain and the range of the quadratic-like restriction of g guaranteed by the Main Theorem ($\mathrm{mod}(V \setminus U) > \mu = \mu(\kappa)$), then we have the following. In the simplest case, when the renormalization $\mathcal{R}(\mathcal{R}^{n+i}f)$ involves going through a long saddle-node cascade only once, then $\mathcal{R}^{n+i+1}f$ has a quadratic-like restriction with the domains sufficiently close to U and V, so that

$$\mathrm{mod}\,\mathcal{R}^{n+i+1}f > \mu/2.$$

Otherwise, the quadratic-like restriction $g : U \to V$ approximates a first return map in the principal nest of $\mathcal{R}^{n+i}f$; in which case, $\mathcal{R}^{n+i+1}f$ has a quadratic-like restriction with an even smaller domain (that is, the range of this restriction is close to V, and the domain is close to a proper subdomain of U). Again, we have

$$\mathrm{mod}\,\mathcal{R}^{n+i+1}f > \mu/2.$$

Therefore, at most every N levels, the sequence $\mathcal{R}^n f$ generates a definite modulus. For the intermediate levels, the lower bound on the modulus follows from compactness of $\mathcal{E}_\sigma$. $\square$

References

[GS] J. Graczyk, G. Swiatek. Polynomial-like property for real quadratic polynomials. Topology Proc. 21 (1996), 33–112.

[Ep] A. Epstein, Towers of finite type complex analytic maps. PhD Thesis, CUNY, 1993.

[EKT] A. Epstein, L. Keen, C. Tresser. The set of maps $F_{a,b} : x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any given rotation interval is contractible. *Commun. Math. Phys.* **173**, 313-333, 1995.

[EY] A. Epstein, M. Yampolsky. The universal parabolic map. *Erg. Theory and Dynam. Systems*, to appear.

[Hin] B. Hinkle. Parabolic limits of renormalization. Ergodic Theory Dynam. Systems 20 (2000), no. 1, 173–229.

[LvS] G. Levin, S. van Strien. Local connectivity of the Julia set of real polynomials. Ann. of Math. (2) 147 (1998), no. 3, 471–541.

[Lyu3] M. Lyubich. Combinatorics, geometry and attractors of quasi-quadratic maps. Ann. of Math. (2) 140 (1994), no. 2, 347–404.

[Lyu4] M.Lyubich. Dynamics of quadratic polynomials, I-II. Acta Math., v. 178 (1997), 185-297.

[Lyu5] M. Lyubich. Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture. *Ann. of Math.* (2) **149**(1999), no. 2, 319–420.

[Lyu6] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Annals of Math.*, **156**(2002), no. 1, 1–78.

[LY] M. Lyubich and M.Yampolsky. Dynamics of quadratic polynomials: complex bounds for real maps. *Ann. l'Inst. Fourier* **47**, 4(1997), 1219-1255.

[McM1] C. McMullen. Complex dynamics and renormalization. *Annals of Math. Studies*, v.135, Princeton Univ. Press, 1994.

[McM2] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. *Annals of Math. Studies*, Princeton University Press, 1996.
[MvS] W. de Melo & S. van Strien. *One dimensional dynamics.* Springer-Verlag, 1993.
[Sh] M. Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. Ann. of Math. (2) 147 (1998), no. 2, 225–267.
[Sul1] D.Sullivan. Quasiconformal homeomorphisms and dynamics, topology and geometry. *Proc. ICM-86, Berkeley*, v. II, 1216-1228.
[Sul2] D.Sullivan. Bounds, quadratic differentials, and renormalization conjectures. *AMS Centennial Publications.* **2***: Mathematics into Twenty-first Century* (1992).
[Ya] M. Yampolsky. The attractor of renormalization and rigidity of towers of critical circle maps. *Comm. Math. Phys.* **218**(2001), no. 3, 537-568.

Department of Mathematics, University of Toronto
E-mail address: yampol@math.toronto.edu