ON THE EIGENVALUES OF A RENORMALIZATION OPERATOR

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ABSTRACT. We study the unstable eigenvalues of the renormalization of critical circle maps. The fixed points of this renormalization operator are critical circle maps f_n with rotation numbers $\rho_n = \cfrac{1}{n + \cfrac{1}{n + \cdots}}$. We show that the corresponding unstable eigenvalues $\lambda_n \asymp n^2$.

1. INTRODUCTION

This note is concerned with the unstable eigenvalues of the cylinder renormalization operator $\mathcal{R}_{cyl}$ of critical circle maps [Ya2]. Recall that for every natural number n, there exists a critical circle map f_n with rotation number

$$\rho(f_n) = \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cdots}}},$$

which is fixed under $\mathcal{R}_{cyl}$. The linearization of $\mathcal{R}_{cyl}$ at this fixed point has a single eigenvalue outside of the closed unit disc, which we denote λ_n. Note that the mappings f_n converge to f_∞ which is the fixed point of the parabolic renormalization operator $\mathcal{P}$ studied in [EY]. Although the rigorous proof of the existence of λ_n has been given only recently in [Ya2], the numerical evidence has long existed. The numerical estimates for λ_n have led one to believe that the values of $\lambda_n \to \infty$. In this note we prove a quantitative version of this statement:

Theorem 1.1. *The following asymptotic estimate holds:*

$$\lambda_n \asymp n^2.$$

A technical remark needs to be made here: in the definition of $\mathcal{R}_{cyl}$ given in [Ya2] we fixed the number $N \in \mathbb{N}$ such that $\mathcal{R}_{cyl} f$ corresponds to the N-th renormalization $\mathcal{R}^n f$ in the usual sense. Thus the cylinder renormalization as defind in that paper shifts the continued fraction representing the rotation number by N symbols to the left, rather than by one. The underlying reason for this was to ensure that the iterate $f^{q_{N-1}}$ has a fundamental

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domain which is small enough to fit inside the domain of the renormalized map f. The problem here may only occur when n is small, for when it is sufficiently large, $\mathcal{R}_{cyl}$ is close to a parabolic renormalization operator, and the condition on the domain is satisfied with $N = 1$. Since we are concerned with $n \to \infty$, we will bend the notation throughout this note, using the same notation for the cylinder renormalization as in [Ya2], and letting $N = 1$ in its definition. The reader should keep in mind, however, that what we say is only true for n large enough.

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2. Some preliminaries

Cylinder renormalization operator. Let us recall briefly the cylinder renormalization operator $\mathcal{R}_{cyl}$ as defined in [Ya2]. We refer the reader to that paper for a detailed discussion, we will generally adapt the same notations. Firstly, let us denote $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ the natural projection. For an equatorial topological annulus $U \subset \mathbb{C}/\mathbb{Z}$ denote $\mathbf{A}_U$ the space of bounded analytic maps $\phi : U \to \mathbb{C}/\mathbb{Z}$, such that $\phi(\mathbb{T})$ is homotopic to $\mathbb{T}$, equipped with the uniform metric. To turn this space into a Banach manifold, consider the Banach space $\tilde{\mathbf{A}}_U$ of bounded analytic 1-periodic functions from $\pi^{-1}(U) \to \mathbb{C}$ with the sup norm, and use the local homeomorphism $\tilde{\mathbf{A}}_U \to \mathbf{A}_U$ given by

$$\psi \mapsto \pi \circ (\psi + \mathrm{Id}) \circ \pi^{-1}$$

to define the atlas on $\mathbf{A}_U$. We denote $\mathbf{C}_U$ the codimension two submanifold of $\mathbf{A}_U$ consisting of maps with a cubic critical point at the origin. Let $\mathbf{C}_U^{\mathbb{R}} \subset \mathbf{C}_U$ be the real Banach manifold consisting of the critical circle maps in $\mathbf{C}_U$ (the *real slice* of $\mathbf{C}_U$). As in [Ya2], a tangent space to $\mathbf{C}_U$ will be naturally identified with a Banach subspace $\mathbf{B}_U \subset \tilde{\mathbf{A}}_U$, $\mathbf{B}_U^{\mathbb{R}}$ will again denote the real slice.

Given a critical cylinder map $f \in \mathbf{C}_U$ let us say that it is *cylinder renormalizable*, or simply *renormalizable*, if there exists $k > 1$ and an equatorial annulus $V \subset \mathbb{C}/\mathbb{Z}$ such that following holds:

- there exist repelling periodic points p_1, p_2 of f in U with periods k and a simple arc l connecting them such that $f^k(l)$ is a simple arc, and $f^k(l) \cap l = \{p_1, p_2\}$;
- the iterate f^k is defined and univalent in the domain C_f bounded by l and $f^k(l)$, the corresponding inverse branch $f^{-k}|_{f^k(C_f)}$ univalently extends to C_f; and the quotient of $\overline{C_f \cup f^k(C_f)} \setminus \{p_1, p_2\}$ by the action of f^k is a Riemann surface conformally isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ (we will call a domain C_f with these properties a *fundamental crescent of* f^k);
- for a point $z \in \bar{C}_f$ with $\{f^j(z)\}_{j \in \mathbb{N}} \cap \bar{C}_f \neq \emptyset$, set $R_{C_f}(z) = f^{n(z)}(z)$ where $n(z) \in \mathbb{N}$ is the smallest value for which $f^{n(z)}(z) \in \bar{C}_f$. We further require that there exists

a point c in the domain of R_{C_f} such that $f^m(c) = 0$ for some $m < n(c)$; and if we denote $\hat{f}$ the projection of R_{C_f} to $\mathbb{C}/\mathbb{Z}$ with $c \mapsto 0$, then $\hat{f} \in \mathbf{C}_V$.

We will say that the new critical circle map $\hat{f}$ is a *cylinder renormalization* of f with period k.
We will see below (Lemma 2.1) that every analytic critical circle map sufficiently close to a parabolic map is cylinder renormalizable. It is shown in [Ya2] that there exists an equatorial annulus $U \subset \mathbb{C}/\mathbb{Z}$ such that the cylinder renormalization extends to a compact analytic operator $\mathcal{R}_{cyl}$ from an open subset of $\mathbf{C}_U$ to $\mathbf{C}_U$ which preserves $\mathbf{C}_U^{\mathbb{R}}$. For $n \in \mathbb{N} \cup \{\infty\}$ we will denote $\mathbf{S}_n$ the slice of the space of critical circle maps specified by the condition

$$\mathbf{S}_n = \{f \in \mathbf{C}_U^{\mathbb{R}} |\ \rho(f) = \frac{1}{n + \cdots}\}.$$

The operator $\mathcal{R}_{cyl}$ is analytic on each of the slices $\mathbf{S}_n$, $n \neq \infty$. It is shown in [Ya2], that there is a fixed poinr f_n of $\mathcal{R}_{cyl}$ in each of $\mathbf{S}_n$, which is hyperbolic with a one-dimensional unstable direction. As in [EY] we denote $\mathbf{P} \subset \mathbf{S}_\infty$ the limit of $\mathbf{S}_n$ as $n \to \infty$.

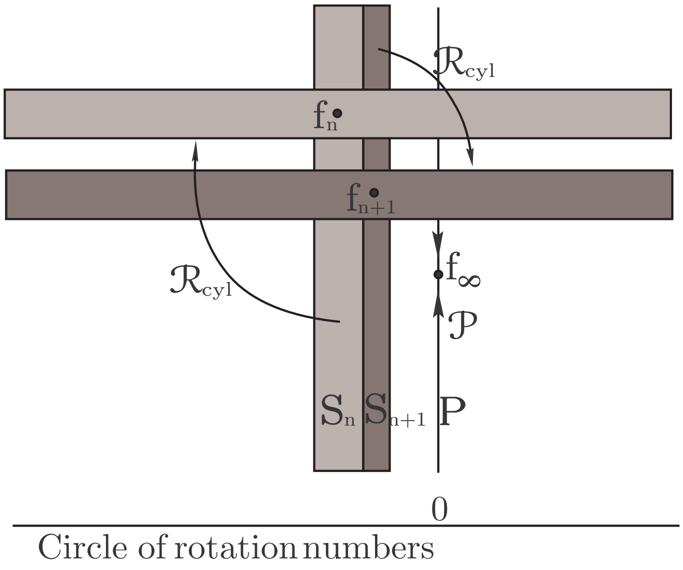

FIGURE 1

Parabolic bifurcation. Let us briefly recap below some facts of the theory of parabolic bifurcation of analytic germs, as applied, in particular, to critical circle maps. Douady's expository article [Do] serves as a good introduction into the theory, a more detailed account may be found in [Sh]. For a map $g_\infty \in \mathbf{P}$ having a fixed point $p \in \mathbb{T}$ with unit eigenvalue we will denote $\Phi^A : U^A \to \mathbb{C}$ and $\Phi^R : U^R \to \mathbb{C}$ the attracting and repelling Fatou coordinates,

C^A and C^R the corresponding Fatou cylinders, and $\pi_A : U^A \to C^A$ and $\pi_R : U^R \to C^R$ the natural projections. Any conformal *transit homeomorphism* $\tau : C^A \to C^R$ is a translation in suitable coordinates. Lifting it produces a map $\bar{\tau} : U^A \to \mathbb{C}$ satisfying

$$\tau \circ \pi_A = \pi_R \circ \bar{\tau}.$$

We will sometimes write $\tau \equiv \tau_\theta$, and $\bar{\tau} = \bar{\tau}_\theta$, where

$$\Phi^R \circ \bar{\tau} \circ (\Phi^A)^{-1}(z) \equiv z + \theta \operatorname{mod}(z).$$

Suppose for a map $g \in \mathbf{C}_U$ in a sufficiently small neighborhood of g_∞ the parabolic point splits into a complex conjugate pair of repelling fixed points $p_g \in \mathbb{H}$ and $\bar{p}_g$ with multipliers $e^{2\pi i \pm \alpha(g)}$. In this situation one may still speak of attracting and repelling petals:

Lemma 2.1 (**Douady Coordinates**). *There exists a neighborhood $U(g_\infty) \subset \mathbf{C}_U$ of the map g_∞ such that for any $g \in U(g_\infty)$ with $|\arg \alpha(g)| < \pi/4$, there exist topological discs U^A_g and U^R_g whose union is a neighborhood of p, and injective analytic maps*

$$\Phi^A_g : U^A \to \mathbb{C} \text{ and } \Phi^R_g : U^R_f \to \mathbb{C}$$

unique up to post-composition by translations, such that

$$\Phi^A_g(g(z)) = \Phi^A_g(z) + 1 \text{ and } \Phi^R_g(g(z)) = \Phi^R_g(z) + 1.$$

The quotients $C^A_g = U^A_g/g$ and $C^R_g = U^R_g/g$ are Riemann surfaces conformally equivalent to $\mathbb{C}/\mathbb{Z}$.

An arbitrary choice of real basepoints $a \in U^A$ and $r \in U^R$ enables us to specify the Fatou and Douady coordinates uniquely by requiring that they turn to 0 at the corresponding basepoint. In this way we obtain well-defined branches ${}_a\Phi^A(a) = {}_a\Phi^A_g(a) = 0$, and ${}_r\Phi^R(r) = {}_r\Phi^R_g(r) = 0$. The following fundamental theorem first appeared in [DH1]:

Theorem 2.2. *Let $a(g)$, $r(g)$ be continuous functions on a neighborhood g_∞ with $a(g_\infty) = a$, $r(g_\infty) = r$. Then the maps ${}_{a(g)}\Phi^A_g$, ${}_{r(g)}\Phi^R_g$ depend continuously on g with respect to the compact-open topology, and*

$$_{a(g)}\Phi^A_g \to {}_a\Phi^A \text{ and } {}_{r(g)}\Phi^R_g \to {}_r\Phi^R$$

uniformly on compact subsets of U^A and U^R respectively.
Moreover, select the smallest $n(g) \in \mathbb{N}$ for which $g^{n(g)}(a) \geq r$. Then

$$g^{n(g)}(z) = \left({}_{r(g)}\Phi^R_g\right)^{-1} \circ T_{\theta(g)+K} \circ {}_{a(g)}\Phi^A_g$$

wherever both sides are defined. In this formula $T_\alpha(z)$ denotes the translation $z \mapsto z + \alpha$, $\theta(g) \in \mathbb{T}$ is given by

$$\theta(g) = 1/\alpha(g) + \underset{\alpha(g)\to\infty}{o(1)} \mod 1,$$

and the real constant K is determined by the choice of the basepoints a, r. Thus for a sequence $\{g_k\} \subset U(g)$ converging to g, the iterates $g_k^{n(g_k)}$ converge locally uniformly if and only if there is a convergence $\theta(g) \to \theta$, and the limit in this case is a certain lift of the transit homeomorphism τ_θ for the parabolic map g_∞.

For a map $g_\infty \in \mathbf{P}$ denote $R_{g_\infty} : C^R \to C^A$ the projection of the first return map from $U^R \to U^A$. We set

$$\mathcal{P}_\theta(g_\infty) = R_{g_\infty} \circ \tau_\theta,$$

and further denote

$$\mathcal{P}(g_\infty) = \mathcal{P}_{\theta(g_\infty)}(g_\infty),$$

where $\theta(g_\infty)$ is chosen in such a way that $\mathcal{P}(g_\infty) \in \mathbf{P}$. The transformation $\mathcal{P} : \mathbf{P} \to \mathbf{P}$ is the *parabolic renormalization operator*, studied in [EY]; it is shown in that paper to be an analytic operator in a Banach submanifold of $\mathbf{C}_U$ with a globally attracting fixed point $f_\infty = \lim f_n$. Lemma 2.1 in conjunction with Theorem 2.2 imply:

Proposition 2.3. *For each n sufficiently large, a map $g \in \mathbf{S}_n$ is cylinder renormalizable with period 1. Moreover, if $g_n \in \mathbf{S}_n$ are such that $g_n \to g_\infty$, and $\mathcal{R}_{cyl}(g_n) \to h$, then*

$$\mathcal{P}_\theta(g_\infty) = h$$

for some $\theta \in \mathbb{T}$.

A schematic description of the action of $\mathcal{R}_{cyl}$ and $\mathcal{P}$ is seen in Figure 1.
Let us now make the following definition. Let $R_\epsilon : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ denote the quotient $T_\epsilon \bmod \mathbb{Z}$ and set

$$g_\epsilon \equiv R_\epsilon \circ g_\infty, \text{ for } |\operatorname{Arg}(\epsilon)| < \frac{\pi}{4}.$$

For $|\epsilon| < \epsilon_0$ small enough the map g_ϵ satisfies the conditions of the existence of Douady coordinates. Let us select points $a \in U^A$ and $r \in U^R$ such that $g_\infty^{n_1}(0) = a$ and $g_\infty^{n_2}(r) = 0$, and set $g_\epsilon^{n_1}(0) = a(\epsilon)$, $g_\infty^{-n_2}(0) = r(\epsilon)$ (the latter denotes the branch of the inverse, which continuously depends on ϵ and produces r when $\epsilon = 0$). With this choice of basepoints let

$$\theta(\epsilon) = \theta(g_\epsilon),$$

as in Theorem 2.2. The quantity $\theta(\epsilon)$ is a version of the *phase* defined in [Do]. By the standard considerations (see [Do] and [Sh]) we have:

Theorem 2.4. *The function $\theta(\epsilon)$ is analytic. Moreover, it can be lifted to $\tilde{\theta}(\epsilon)$ with values in $\mathbb{R}$ and*

$$\tilde{\theta}(\epsilon) = \frac{\text{const}}{\epsilon} + \underset{\epsilon \to 0}{O(1)}.$$

3. Proof of Theorem 1.1

Let us set $g_\infty \equiv f_\infty$ and let g_ϵ, $\theta(\epsilon)$ be as above. Let us set

$$\Gamma = \{g_\epsilon\},\ \Gamma^{\mathbb{R}} = \Gamma \cap \mathbf{C}_U^{\mathbb{R}} = \{g_\epsilon |\ \epsilon \in \mathbb{R}\}.$$

As follows from the analysis of [Ya3], the family $\Gamma^{\mathbb{R}}$ is a global transversal to the stable foliation of $\mathcal{R}_{cyl}$ in $\mathbf{C}_U^{\mathbb{R}}$. Let us consider the interval

$$I_n = \{\epsilon \in \mathbb{T} |\ g_\epsilon \in \cap \mathbf{S}_n\}.$$

The critical point 0 is periodic under g_ϵ for $\epsilon \in \partial I_n$. Hence, the preimage

$$\tilde{\theta}^{-1}(I_n) = [-n-K, -n-K+1]$$

for some $K \in \mathbb{N}$ depending on the choice of the lift. Now Theorem 2.4 implies that

$$|I_n| \asymp \frac{1}{n^2}.$$

Select an analytic projection π from a neighborhood of $\mathbf{C}_U^{\mathbb{R}}$ in $\mathbf{C}_U$ to the curve Γ which is transverse to the unstable direction of $\mathcal{R}_{cyl}$. Denote

$$\Psi_n \equiv \pi \circ \mathcal{R}_{cyl} : \Gamma \cap \mathbf{S}_n \to \Gamma,$$

and define ψ_n by

$$\Psi_n(g_\epsilon) = g_{\psi_n(\epsilon)}.$$

Compactness considerations imply that there exists an equatorial annulus $V \subset \mathbb{C}/\mathbb{Z}$ independent of n, such that ψ_n conformally extends to V. By the Koebe Distortion Theorem, $|D\psi_n(z)|_{\mathbb{T}}| \asymp 1$. Hence,

$$|D\Psi_n|_{\mathbb{T}}| \asymp \frac{1}{n^2},$$

and since

$$|D\Psi_n|_{\mathbb{T}}| \asymp |D\mathcal{R}_{cyl}|_{W^u(f_n)}|$$

the claim of the theorem follows.

4. Concluding remarks.

Firstly, let us note that a parallel result can be proved in the same way for quadratic-like renormalization of maps with essentially bounded combinatorics (see [Hin], [LY], [Ya4]). The combinatorial description of the relevant sequences of renormalization fixed points is somewhat invloved, so we will not attempt to give it here. The corresponding version of the estimate on the eigenvalues in this case is $\lambda_n \asymp n^3$. The difference in the exponent is due to the change in the estimate in Theorem 2.4 from const$/\epsilon$ to const$/\sqrt{\epsilon}$; which is in turn due to the difference in the order of the critical point of the dynamical systems. For a particular case of essentially bounded combinatorics, this estimate follows from the analysis of [BDDS].

Another point of interest is related to the quadratic-like renormalization of maps of satellite type. Let $\mathcal{M}$ denote the Mandelbrot set, and let $\mathcal{M}_\lambda$ denote its preimage under the branched double-covering map $\kappa : \lambda \mapsto c = \lambda/2 - \lambda^2/4$. The value of $\lambda(c)$ is equal to the multiplier of a fixed point of the polynomial $f_c(z) = z^2 + c$. The p/q-root is by definition the point $r_{p/q} \in \partial\mathcal{M}$ which corresponds to the λ-value $e^{2\pi i p/q}$. The satellite copy $\mathcal{M}_{p/q}$ is the small copy of the Mandelbrot set, which grows out of $r_{p/q}$, the p/q-limb $\mathcal{L}_{p/q}$ is the connected component of $\mathcal{M} \setminus \{r_{p/q}\}$ which contains $\mathcal{M}_{p/q}$. It is conjectured, but not as of yet proved, that the renormalization operator associated to each $\mathcal{M}_{p/q}$ has a hyperbolic fixed point with a single unstable direction. K. Khanin informed us that in 1983 in [GSK] it was conjectured from numerical experiments that if $\lambda_{p/q}$ is the unstable eigenvalue of that fixed point, then

$$|\lambda_{p/q}| \asymp q^2.$$

Several years later, Cvitanović and Myrheim [CM] conjectured that

$$\frac{\lambda_{p/q}}{q^2 \exp(2i\pi(p/q - 1/2))} \underset{q\to\infty}{\longrightarrow} 1.$$

The considerations of parabolic bifurcation similar to the ones employed in this paper give $\operatorname{diam} \kappa^{-1}(\mathcal{M}_{p/q}) \sim q^2$, which is consistent with the above. It is also worth mentioning in this regard the long-standing conjecture on the size of the satellite limbs of $\mathcal{M}$ due to Douady, Hubbard, and others:

Conjecture. *If $\mathcal{L}'_{p/q}$ denotes the component of the preimage $\kappa^{-1}(\mathcal{L}_{p/q})$ attached to the unit disk, then*

$$\operatorname{diam} \mathcal{L}'_{p/q} \asymp 1/q^2.$$

We conclude with a table of values of the first few λ_n's graciously provided to us by Keith Briggs of BTexact Technologies:

n	λ_n	λ_n/n^2
1	2.83365302	2.83365302
2	6.79922494	1.69980623
3	13.76028148	1.52892016
4	24.62034614	1.53877163
5	40.38691319	1.61547652

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