MAT 137Y: Calculus with proofs
Assignment 10
Due on Thursday, April 8 by 11:59pm via Crowdmark

Instructions:

- You will need to submit your solutions electronically via Crowdmark. See MAT137 Crowdmark help page for instructions. Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.

- You may submit individually or as a team of two students. See the link above for more details.

- You will need to submit your answer to each question separately.

- This problem set is based on roughly the first half of Unit 14.

1. Construct a power series whose interval of convergence is exactly $[0, 3]$.

   Note: This should go without saying, but it is not enough to write down the power series: you also need to *prove* that its interval of convergence *is* $[0, 3]$.

2. Is it possible for a power series to be conditionally convergent at two different points? Prove it.

3. Let $f(x) = x^{100} e^{2x^5} \cos(x^2)$. Calculate $f^{(137)}(0)$.

   You may leave your answer indicated in terms of sums, products, and quotients of rational numbers and factorials, but not in terms of sigma notation. For example, it is okay to leave your answer as
   $$15 + \frac{4!}{10} - \frac{2 \cdot 7!}{13 \cdot 13!}$$
   Notice that this is just an example, not the actual answer.

   Hint: Use Taylor series.

4. Let I be an open interval. Let f be a function defined on I. Let $a \in I$. Assume f is C^2 on I. Assume $f'(a) = 0$. As you know, a is a candidate for a local extremum for f. The "2nd Derivative Test" says that, under these circumstances:

   - IF $f''(a) > 0$, THEN f has a local minimum at a.
   - IF $f''(a) < 0$, THEN f has a local maximum at a.

   This theorem is easier to justify, and to generalize, using Taylor polynomials.

(a) Let P_2 be the 2nd Taylor polynomial for f at a. Write an explicit formula for P_2.

Using the idea that $f(x) \approx P_2(x)$ when x is close to a, write an intuitive explanation for the 2nd Derivative Test.

Note: We are not asking for a proof yet. Rather, we are asking for a short, simple, handwavy argument that would convince an average student that this result "makes sense" and "seems true".

(b) Now write an actual, rigorous proof for the 2nd Derivative Test. You will need to use the first definition of Taylor polynomial (the one in terms of the limit), the definition of limit, and the definition of local extremum.

Note: There are many other ways to prove the 2nd Derivative Test, but we want you to do it specifically this way. It will help with the next questions.

(c) What happens if $f''(a) = 0$? In that case, we look at $f^{(3)}(a)$; if it is also 0, then we look at $f^{(4)}(a)$, and we keep looking till we find one derivative that is not 0 at a.

More specifically, assume that f is C^n at a for some natural number $n \geq 2$ and that $f^{(n)}(a)$ is the smallest derivative that is not 0 at a. In other words, $f^{(k)}(a) = 0$ for $1 \leq k < n$ but $f^{(n)}(a) \neq 0$.

Using the same ideas as in Question 4a, complete the following statements, and give an intuitive explanation for them:

- IF ..., THEN f has a local minimum at a.
- IF ..., THEN f has a local maximum at a.
- IF ..., THEN f does not have a local extremum at a.

When you complete this, you will have come up with a new theorem. Let's call it the "Beyond-the-2nd-derivative Test". Make sure your theorem takes care of all possible cases.

(d) [**Do not submit.**] Imitate the proof in 4b to prove the Beyond-the-2nd-derivative Test.

Note: Why did we ask you to write both a "simple, intuitive explanation" and a "formal proof"? Because that is how research in mathematics often happens. The intuitive explanation is how we come up with conjectures and discover new theorems (as in Questions 4a and 4c). After that, the formal proof is how we make sure the conjecture is not merely a conjecture but a true theorem (as in Questions 4b and 4d). We need both.