MAT 137Y: Calculus with proofs
Assignment 10 - Sample Solutions

1. Construct a power series whose interval of convergence is exactly $[0, 3]$.

   Note: This should go without saying, but it is not enough to write down the power series: you also need to *prove* that its interval of convergence *is* $[0, 3]$.

Solution:

Consider the series $S(x) = \sum_{n=1}^{\infty} \frac{(x-1.5)^n}{n^2 \cdot 1.5^n}$.

I will prove that its interval of convergence is exactly $[0, 3]$.

First, I will try to use the Ratio Test. The series is convergent at the center ($x = 1.5$). If $x \neq 1.5$, to use the Ratio Test, I compute the following limit:

$$L(x) = \lim_{n\to\infty} \frac{\left|\frac{(x-1.5)^{n+1}}{(n+1)^2 \cdot 1.5^{n+1}}\right|}{\left|\frac{(x-1.5)^n}{n^2 \cdot 1.5^n}\right|} = \lim_{n\to\infty} \left[\frac{n^2}{(n+1)^2} \frac{|x-1.5|}{1.5}\right] = \frac{|x-1.5|}{1.5}$$

According to the Ratio Test

- If $L(x) < 1$, then the series $S(x)$ is absolutely convergent. Notice that, for $x \neq 1.5$:

  $$L(x) < 1 \iff \frac{|x-1.5|}{1.5} < 1 \iff |x-1.5| < 1.5 \iff 0 < x < 3$$

  We had to exclude $x = 1.5$ from the calculation of $L(x)$, but we should include it in the interval of convergent. Therefore, the interval of convergence contains at least $(0, 3)$.

- If $L(x) > 1$, then the series $S(x)$ is divergent. Notice that

  $$L(x) > 1 \iff \frac{|x-1.5|}{1.5} > 1 \iff |x-1.5| > 1.5 \iff (x < 0 \text{ or } x > 3)$$

It remains to analyze what happens at the end points: $x = 0$ and $x = 3$:

- $S(3) = \sum_{n=1}^{\infty} \frac{(3-1.5)^n}{n^2 \cdot 1.5^n} = \sum_{n=1}^{\infty} \frac{1.5^n}{n^2 \cdot 1.5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which I know is convergent (It is a p-series with $p = 2$).
- $S(0) = \sum_{n=1}^{\infty} \frac{(0-1.5)^n}{n^2 \cdot 1.5^n} = \sum_{n=1}^{\infty} \frac{(-1.5)^n}{n^2 \cdot 1.5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which I know is convergent (We have two reasons: it is absolutely convergent, or use the alternating series test).

Putting it all together, I have proven the interval of convergence of $S(x)$ is $[0, 3]$, including both endpoints.

How did I come up with this solution?

I think it is easier to first find a power series whose interval of convergence includes both endpoints. We know that the geometric series

$$\sum_{n}^{\infty} x^n$$

has interval of convergence $(-1, 1)$. A little bit of trial error may lead us to consider

$$T(x) \;=\; \sum_{n}^{\infty} \frac{x^n}{n^2}$$

and discover that it has interval of convergence $[-1, 1]$. In other words:

$$T(x) \text{ is convergent } \iff |x| \leq 1.$$

From here, we have to "shift" the power series (so that it is centered at 1.5 rather than 0) and then "stretch" it (so that the radius of convergence is 1.5 rather than 1). This leads us to the power series

$$S(x) \;=\; \sum_{n}^{\infty} \frac{1}{n^2} \left(\frac{x - 1.5}{1.5} \right)^n$$

and we notice that

$$S(x) \text{ is convergent } \quad \iff \quad \left| \frac{x - 1.5}{1.5} \right| \leq 1 \quad \iff \quad |x - 1.5| \leq 1.5 \quad \iff \quad 0 \leq x \leq 3$$

2. Is it possible for a power series to be conditionally convergent at two different points? Prove it.

Solution: Yes, it is. Consider the power series

$$S(x) \;=\; \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{2n} \tag{1}$$

Then, $S(1) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
This is the alternating harmonic series, that we know is conditionally convergent:

- It is convergent by the Alternating Series Test, but
- it is not absolutely convergent since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a p series with $p = 1$

In addition, $S(-1) \;=\; \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \;=\; S(1)$, so it is also conditionally convergent.

I have proven that the power series in (1) is conditionally convergent at two different points: at $x = 1$ and at $x = -1$.

Note: Perhaps you may object that (1) is not a power series because it does not look like a series of the form

$$\sum_{m=1}^{\infty} a_m x^m$$

But it is of this form! Just define $a_m \;=\; \begin{cases} 0 & \text{if } m \text{ is odd} \\ (-1)^n/n & \text{if } m = 2n \text{ for some } n \in \mathbb{Z}^+ \end{cases}$

How did I come up with this solution?

We know that a power series is absolutely convergent in the interior of its interval of convergence, and divergent in the exterior. The only points that are candidates to make a power series conditionally convergent are the endpoints of the interval of convergence. Can I construct a power series that is conditionally convergent at *both* endpoints? Let's say the power series is

$$S(x) = \sum_{m}^{\infty} a_m x^m$$

and the radius of convergence is R. Then I would like for both of these power series

$$S(R) = \sum_{m}^{\infty} a_m R^m, \qquad S(-R) = \sum_{m}^{\infty} a_m (-R)^m = \sum_{m}^{\infty} (-1)^m a_m R^m$$

to be conditionally convergent. Let's pay attention to the signs.

- If $a_m > 0$ for all m, the series $S(R)$ is a positive series. It can only be absolutely convergent or divergent, but not conditionally convergent. The same happens if $a_m < 0$ for all m.
- Perhaps we should make the sequence $\{a_m\}$ alternating in sign? Something like $a_m = (-1)^m b_m$ with $b_m > 0$ for all m? No, that won't work either! In that case $S(-R)$ will be a positive series, and we have the same problem.
- We need a more complicated sign pattern for $\{a_m\}$. Perhaps something like

  $$+, +, -, -, +, +, -, -, \ldots$$

  That could work. Or perhaps, we can use a simpler pattern:

  $$0, +, 0, -, 0, +, 0, -, \ldots$$

  Aha! I should be looking at a power series that only has even terms, which alternate in sign, something like

  $$\sum_{n}^{\infty} (-1)^n c_n \, x^{2n}$$

  where $c_n > 0$ for all n. Once I know to look at power series like this one, it is easier to come up with a good idea.

There are other solutions, of course. For example, the power series $\displaystyle\sum_{n}^{\infty} \frac{\sin n}{n} x^n$ is also conditionally convergent at $x = 1$ and at $x = -1$, but it is *very* difficult to prove it.

3. Let $f(x) = x^{100}e^{2x^5}\cos(x^2)$. Calculate $f^{(137)}(0)$.

You may leave your answer indicated in terms of sums, products, and quotients of rational numbers and factorials, but not in terms of sigma notation.

Solution:

f is a product of compositions of analytic functions, so it is analytic. In order to find $f^{(137)}(0)$, I want to first find its Maclaurin series. Actually, I need a little bit less. If the Maclaurin series of f is given by

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

then we know that $\boxed{f^{(137)}(0) = 137! \cdot c_{137}}$. Therefore, **I only need to find the coefficient of x^{137} in the Maclaurin series of f.**

We know that, for every $t \in \mathbb{R}$:

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \qquad \cos t = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} t^{2m}$$

Therefore

$$f(x) = x^{100}e^{2x^5}\cos(x^2) = x^{100} \cdot \left[\sum_{k=0}^{\infty} \frac{2^k}{k!} x^{5k}\right] \cdot \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{4m}\right]$$

We know that in the interior of the interval of convergence (in this case all of $\mathbb{R}$) we can multiply power series "term by term", just like we multiply polynomials. Hence

$$f(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left(\frac{2^k}{k!} \cdot \frac{(-1)^m}{(2m)!} x^{100+5k+4m}\right)$$

Fortunately, we do not need to work out all the terms in the sum. We only need the terms that will contribute to c_{137}. In other words, we only need the terms with x^{137}. More specifically, c_{137} will be the *sum* of all the terms of the form

$$\frac{2^k}{k!} \cdot \frac{(-1)^m}{(2m)!} \qquad \text{for } m, k \in \mathbb{N} \text{ satisfying } 100 + 5k + 4m = 137$$

A little bit of algebra shows that there are only two such terms: $(k, m) = (1, 8)$ and $(k, m) = (5, 3)$. Therefore

$$c_{137} = \frac{2^1}{1!} \cdot \frac{(-1)^8}{16!} + \frac{2^5}{5!} \cdot \frac{(-1)^3}{6!} = \frac{2}{16!} - \frac{32}{5! \cdot 6!}$$

and

$$\boxed{f^{(137)}(0) = 137! \cdot \left[\frac{2}{16!} - \frac{32}{5! \cdot 6!}\right]}$$

4. Let I be an open interval. Let f be a function defined on I. Let $a \in I$. Assume f is C^2 on I. Assume $f'(a) = 0$. As you know, a is a candidate for a local extremum for f. The "2nd Derivative Test" says that, under these circumstances:

- IF $f''(a) > 0$, THEN f has a local minimum at a.
- IF $f''(a) < 0$, THEN f has a local maximum at a.

This theorem is easier to justify, and to generalize, using Taylor polynomials.

(a) Let P_2 be the 2nd Taylor polynomial for f at a. Write an explicit formula for P_2. Using the idea that $f(x) \approx P_2(x)$ when x is close to a, write an intuitive explanation for the 2nd Derivative Test.

Note: We are not asking for a proof yet. Rather, we are asking for a short, simple, handwavy argument that would convince an average student that this result "makes sense" and "seems true".

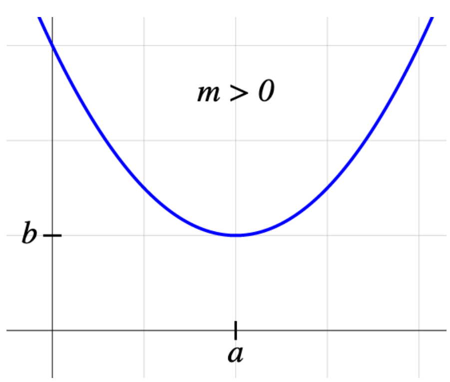
Solution:

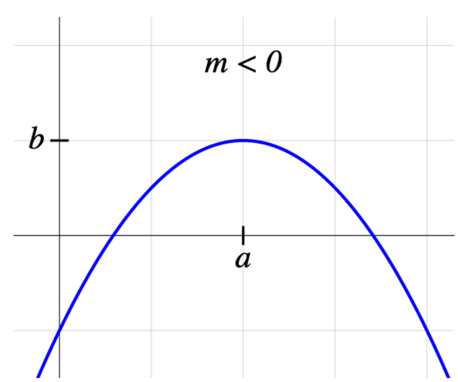
In general, $P_2(x) \;=\; f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2$.
Since $f'(a) = 0$ we can simplify it as

$$\boxed{P_2(x) \;=\; f(a) \;+\; \frac{f''(a)}{2}(x-a)^2} \tag{2}$$

When x is close to a, $f(x) \approx P_2(x)$.
To determine what happens to f at a, I will study what happens to P_2 instead.

We know that the equation of a parabola with vertex (a, b) is $\boxed{y = b + m(x-a)^2}$, where m is a non-zero constant that measures "how much the parabola opens":





- When $m > 0$, the parabola "opens up" and it has a local minimum at $x = a$.
- When $m < 0$, the parabola " opens down" and it has local maximum at $x = a$.

Looking back at Equation (2), we see that $y = P_2(x)$ is the equation of a parabola with $m = \dfrac{f''(a)}{2}$. This explains the 2nd Derivative Test.

(b) Now write an actual, rigorous proof for the 2nd Derivative Test. You will need to use the first definition of Taylor polynomial (the one in terms of the limit), the definition of limit, and the definition of local extremum.

Note: There are many other ways to prove the 2nd Derivative Test, but we want you to do it specifically this way. It will help with the next questions.

Solution:

Let P_2 be the 2nd Taylor polynomial for f at a, as in Question 4a. Let R_2 be the corresponding remainder. Then we can write $f(x) = P_2(x) + R_2(x)$ or, equivalently

$$f(x) \;=\; f(a) + \frac{f''(a)}{2}(x-a)^2 + R_2(x).$$

If $x \neq a$, I can rewrite it as

$$f(x) - f(a) \;=\; (x-a)^2\, h(x) \tag{3}$$

where I define the function h as

$$h(x) \;=\; \frac{f''(a)}{2} + \frac{R_2(x)}{(x-a)^2}$$

Let's assume first that $f''(a) > 0$. I want to prove that f has a local minimum at $x = a$. By definition, what I want to show is

$$\exists \delta > 0,\ \forall x \in \mathbb{R}, \qquad |x-a| < \delta \implies f(x) - f(a) \geq 0.$$

Notice that when $x = a$ we are guaranteed that $f(x) \geq f(a)$ (they are, in fact, equal), so it is enough to show that

$$\exists \delta > 0,\ \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies f(x) - f(a) \geq 0. \tag{4}$$

In order to show this, I am going to prove, instead, that

$$\exists \delta > 0,\ \forall x \in \mathbb{R}, \quad 0 < |x-a| < \delta \implies h(x) \geq 0. \tag{5}$$

Then, (4) will follow simply from (3) and (5).

By definition of Taylor polynomial, we know that

$$\lim_{x\to a} \frac{R_2(x)}{(x-a)^2} \;=\; 0.$$

Hence, using the limit law for sums:

$$\lim_{x\to a} h(x) \;=\; \lim_{x\to a}\left[\frac{f''(a)}{2} + \frac{R_2(x)}{(x-a)^2}\right] \;=\; \frac{f''(a)}{2} \tag{6}$$

Intuitively, since $f''(a) > 0$, (6) tells us that when x is near a, $h(x) > 0$, which is what I need to show. Now I just need to make this intuitive argument rigorous.[1]

[1]Perhaps this feels familiar. We need to show that if a function has positive limit at a point, then it remains positive near the point. This is quite similar to the proof of Q2 in Test 1 - Part B.

- Let us take $\varepsilon = \frac{f''(a)}{2} > 0$ in the definition of $\lim_{x \to a} h(x) = \frac{f''(a)}{2}$.

  Then, by definition of limit, we know that for this value of ε, $\exists \delta > 0$ such that

$$\forall x \in \mathbb{R}, \quad 0 < |x - a| < \delta \implies \left| h(x) - \frac{f''(a)}{2} \right| < \varepsilon \tag{7}$$

  I will show that this value of δ satisfies (5)

- Let $x \in \mathbb{R}$. Assume $0 < |x - a| < \delta$. I need to show that $h(x) \geq 0$.

- From (7) I know that this value of x satisfies

$$h(x) > \frac{f''(a)}{2} - \varepsilon = 0$$

This completes the proof for the case $f''(a) > 0$.

The proof for the case $f''(a) < 0$ is very similar and I won't write the details. The only differences are that we need to use the definition of local maximum instead of local minimum, and that we take $\varepsilon = \left| \frac{f''(a)}{2} \right|$.

(c) What happens if $f''(a) = 0$? In that case, we look at $f^{(3)}(a)$; if it is also 0, then we look at $f^{(4)}(a)$, and we keep looking till we find one derivative that is not 0 at a.

More specifically, assume that f is C^n at a for some natural number $n \geq 2$ and that $f^{(n)}(a)$ is the smallest derivative that is not 0 at a. In other words, $f^{(k)}(a) = 0$ for $1 \leq k < n$ but $f^{(n)}(a) \neq 0$.

Using the same ideas as in Question 4a, complete the following statements, and give an intuitive explanation for them:

- IF ..., THEN f has a local minimum at a.
- IF ..., THEN f has a local maximum at a.
- IF ..., THEN f does not have a local extremum at a.

When you complete this, you will have come up with a new theorem. Let's call it the "Beyond-the-2nd-derivative Test". Make sure your theorem takes care of all possible cases.

Solution: Let P_n be the n-th Taylor polynomial for f at a. Under the given conditions, most of the coefficients of P_n are zero, and we are left with

$$\boxed{P_n(x) \;=\; f(a) + \frac{f^{(n)}(a)}{n!}(x-a)^n} \tag{8}$$

Like in Question 4a, we can say that when x is close to a, $f(x) \approx P_n(x)$.

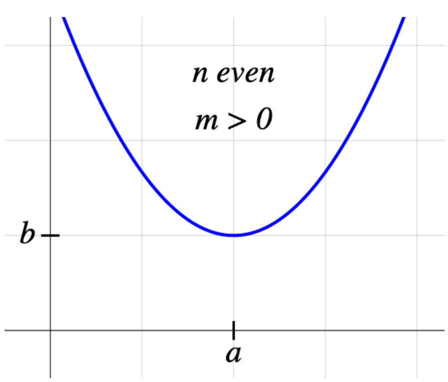
In order to find the local behaviour of f at a, I will study the local behaviour of P_n instead.

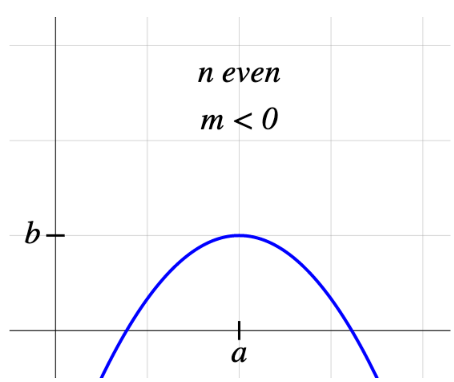
Looking at Equation (8), and remembering how I solved 4a, let's study first the graphs of equations like

$$\boxed{y = b + m(x-a)^n} \tag{9}$$

where $m \neq 0$ and $n \in \mathbb{Z}^+$. What this graph looks like will depend on the parity of n.

- **If n is even**, then $(x-a)^n > 0$ for all $x \neq a$ and this looks similar to a parabola as in Question 4a. The graph will stay above $y = b$ if $m > 0$ and below $y = b$ if $m < 0$

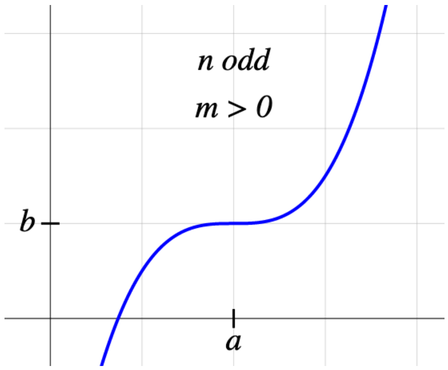


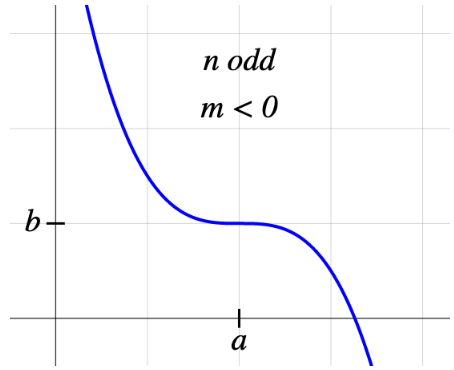


  In other words,

  - If $m > 0$ the graph has a local minimum at $x = a$.
  - If $m < 0$ the graph has a local maximum at $x = a$.

- **If n is odd**, then $(x-a)^n > 0$ for $x > a$, but $(x-a)^n < 0$ for $x < a$. No matter the sign of m, the graph will stay above $y = b$ on one side of a and below $y = b$ at the other side. The graph has an inflection point at $x = a$ rather than a local extremum.





Finally, comparing Equations (8) and (9) we see that $m = \dfrac{f^{(n)}(a)}{n!}$. We can now fill-in the details of the "Beyond-the-2nd-derivate Test:

- IF n is even and $f^{(n)}(a) > 0$, THEN f has a local minimum at a.
- IF n is even and $f^{(n)}(a) < 0$, THEN f has a local maximum at a.
- IF n is odd, THEN f does not have a local extremum at a.