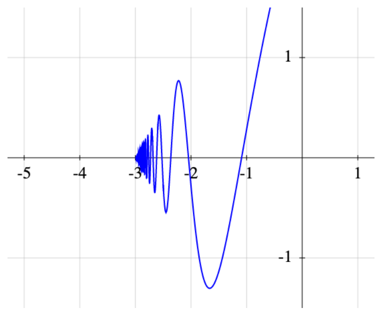
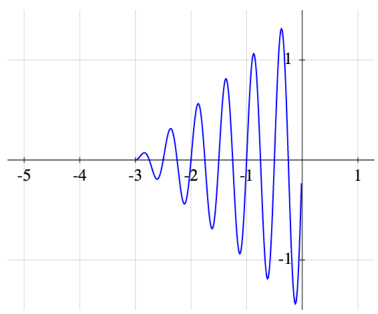
MAT 137Y: Calculus with proofs
Assignment 2 - Comments and common errors

Q1

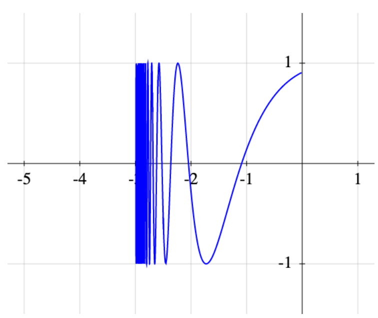
- Surprisingly, some of you did not define $f(0)$, thus failing Condition (a).
- Many of you realized that $f(x)$ must oscillate wildly as $x \to -3^+$. For example, this graph works for that piece of the function:



However, you also need the oscillations to get faster and faster as $x \to -3^+$, so that the function oscillates infinitely many times between positive and negative numbers before reaching -3. Otherwise, you will not get the desire conditions on $\lim_{x\to -3^+} h(h(x))$. For example, this does not work:



In addition, you also need the amplitude of the oscillations to approach 0 as $x \to -3^+$. Otherwise, you won't get $\lim_{x\to -3^+} h(x) = 0$. For example, this does not work:



If you graph was not clear, you could always explain the behaviour with words (like we suggested in the question, and like the sample solutions do).

Q2

- The question asks "Based only on this information, **can you conclude** whether $\lim_{x \to a} [f(x) + g(x)]$ exists or does not exist?" The correct answer is "No, we cannot draw a conclusion one way or another" and the only way to prove this is to provide **two** examples, one with each conclusion. The question is very specific, and it is crystal-clear. There is no ambiguity. We did not ask you

  "Is it possible for the limit to exist?"

  or

  "Can we conclude that the limit does not exist?"

  Rather, we asked you

  "Can we conclude whether the limit exists or does not exist?"

  It is your job to read the question carefully and understand what it asks.

  The correct answer isn't "the limit may exist, and here is one example to prove it", but rather "The limit may or may not exist, and here are two examples to prove it".

Q3

This question was generally done rather well.

Q4

This question was done very poorly. Many of you do not understand proof structure, the difference between a quantified and a fixed variable, or why it matters. Please practice with other similar questions (for example Q7, Q8, and Q10 on the Unit 2 practice problems.)

- A quantified variable has not been fixed or introduced yet. It is just a dummy variable that does not carry any meaning. For example, if you know that

  $$\forall \varepsilon > 0, \ \exists N_2 \in \mathbb{R} \ \text{ such that } \quad x > N_2 \implies |g(x) - L| < \varepsilon \tag{1}$$

  then the variable N_2 does not mean anything. You cannot do algebra with it yet.

  You would need to first say which value of ε you choose, and then say that you fix the value of N_2 in (1) corresponding to that value of ε. Otherwise, N_2 means nothing.

- Conversely, once you have fixed a variable, don't quantify it. For example, after you fix a certain value of N, do not write "$\forall N \in \mathbb{R}, \ldots$".

- Proof structure is important. If you are trying to prove that

  $$\forall M \in \mathbb{R}, \ \exists N \in \mathbb{R}, \ \text{ such that } \ \ldots$$

  you need to first an arbitrary value of M, then tell me what N is (depending on M), and finally verify it works. Otherwise, you are not proving the statement!

- All variables need to be introduced in order. Read the sample solution. Notice it goes in this order:
  - We fix M
  - We use this value of M to explain how we produce N_1 and N_2
  - Then we take $N = \max\{N_1, N_2\}$.

  If you do these steps in a different order, then your proof is meaningless.
- You may not assume that $L > 0$. Why would it be? For example, you cannot take $\varepsilon = L$ as the "epsilon" in the definition of "$\lim_{x\to\infty} g(x) = L$", as it may be negative.
- Notice that in proofs about limits "as $x \to a$" we often write things like "Take $\delta = \min\{\delta_1, \delta_2\}$", but in proofs about limits "as $x \to \infty$" we often write things like "Take $N = \max\{N_1, N_2\}$". If you are memorizing this rather than understanding why, you are doing it wrong, and you will not know what to do when we move to a different kind of proof.