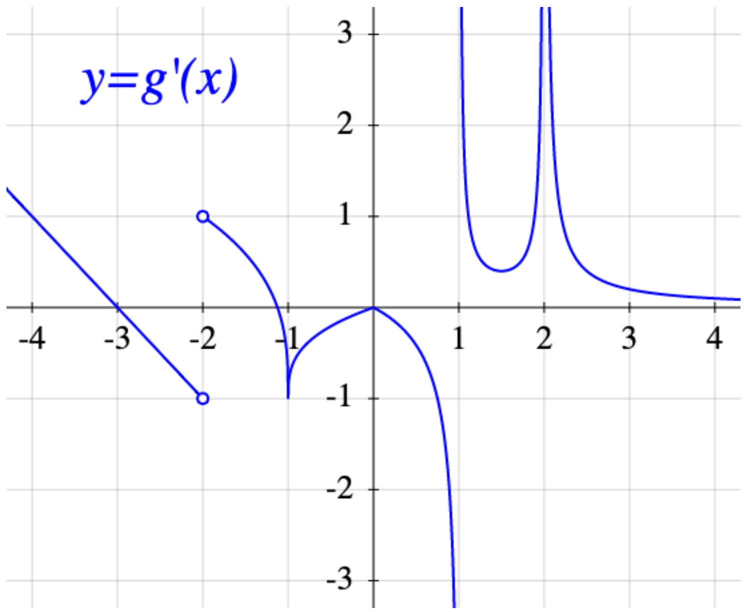
MAT 137Y: Calculus with proofs
Assignment 3 - Sample solutions

Question 1 *The function g has domain $\mathbb{R}$ and is continuous. Below is the graph of its derivative g'. Sketch the graph of g.*



Note: *There could be more than one correct answer.*

Solution: See the graph in Figure 1 below. We are working from a graph rather than an equation, so many of our arguments are intuitive rather than fully rigorous.

The function g is differentiable everywhere except at the points -2, 1, and 2. Note that there is a point close to -1 where g' vanishes: since its value is not specified on the graph, I will call this real number c. Then we have that

1. The graph of g has horizontal tangent lines at -3, c, and 0.
2. The graph of g has tangent lines with positive slope on $(-\infty, -3) \cup (-2, c) \cup (1, 2) \cup (2, \infty)$. Intuitively, g is increasing on these intervals.
3. Similarly, the graph of g has tangent lines with negative slope on $(-3, -2) \cup (c, 0) \cup (0, 1)$. Intuitively, g is decreasing on these intervals.
4. The function $g'(x)$ has a jump discontinuity at -2, with left and right limits equal to -1 and 1, respectively. Since g is continuous there, it is reasonable to conclude that the graph has a corner.[1]

[1]This is technically not our definition of a corner; however, more advanced tools (MVT) can be used to prove that g does indeed satisfy our definition.

5. When x approaches 1 from the left (resp. right), $g'(x)$ approaches $-\infty$ (∞). In addition g is continuous. If we only looked at one side of the graph, it would looked like a vertical tangent line, but g reverts from increasing to decreasing at that point. This is often called a *cusp*.

6. When x approaches 2, the derivative $g'(x)$ approaches ∞. Since g is continuous, the graph has a vertical tangent line at this point.

7. When x approaches $-\infty$, the derivative approaches ∞, so the the lines tangent to the graph of g grow steeper and steeper (with positive slopes) at points further and further to the left.

8. When x approaches ∞, the derivative approaches 0, so the lines tangent to the graph of g grow less and less steep at points further and further to the right, while maintaining a positive slope. Just from the graph of g' it is impossible to tell whether g will have a horizontal asymptote or whether $g(x)$ will approach ∞ as $x \to \infty$ (like, for example, the logarithm function).

9. Notice that at $x = 1$ and at $x = 0$ the graph of g is nice and smooth: the function is differentiable. The function g has inflection points at those values, but that is more advanced that what we know at this moment in the course. It is okay if your graph does not include the inflection points (as long as your function is differentiable at those points).

Since the values of g' at -4 and -1 are easy to read from the graph, I included the tangent lines at the corresponding points.

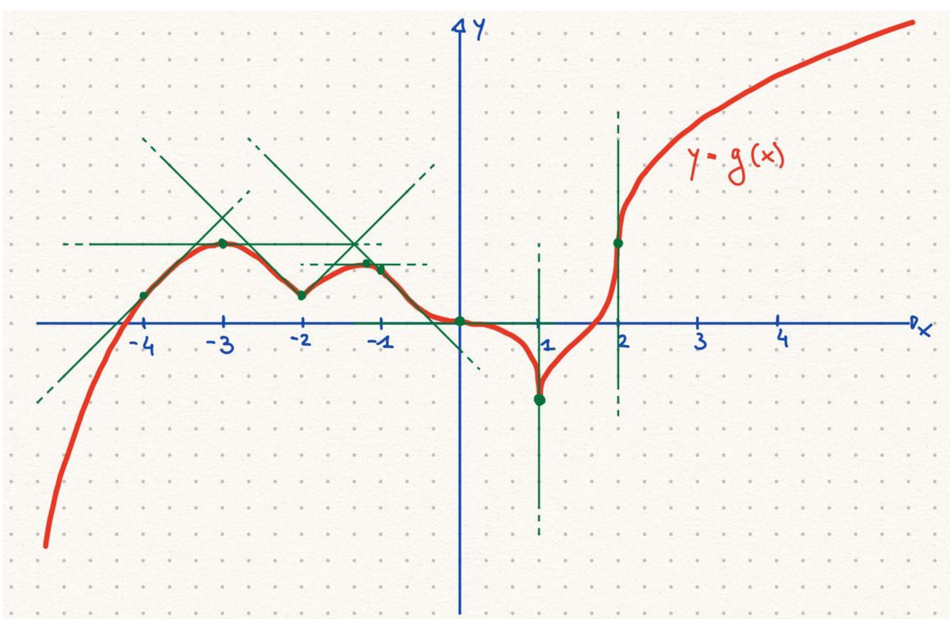


Figure 1: Each green line indicates the tangent at the marked point.

Additional remarks: Notice that, while g' has a cusp at -1 and a corner at 0, these have no immediate significance for the graph of g. Moreover, even if $g'(x)$ approaches 0 when x approaches ∞, this is not enough to conclude whether g has a horizontal asymptote.

Question 2 *Let $a \in \mathbb{R}$. Let f be a function. Assume f is differentiable at a. Assume f is never 0. Let g be the function defined by the equation $g(x) = \dfrac{1}{f(x)}$.*

Prove that g is differentiable at a and that $g'(a) = \dfrac{-f'(a)}{f(a)^2}$.

Write a proof directly from the definition of derivative, without using any of the differentiation rules.

Solution: I want to show that

$$\lim_{h\to 0} \frac{g(a+h)-g(a)}{h} = -\frac{f'(a)}{f(a)^2}\,.$$

Using the definition of $g(x)$, for every $h \in \mathbb{R}$ with $h \neq 0$, I can write

$$\begin{aligned}\frac{g(a+h)-g(a)}{h} &= \frac{1}{h}\cdot\left(\frac{1}{f(a+h)} - \frac{1}{f(a)}\right) = \\ &= \frac{1}{h}\left(\frac{f(a)-f(a+h)}{f(a+h)f(a)}\right) = \\ &= -\frac{1}{f(a+h)f(a)}\cdot\frac{f(a+h)-f(a)}{h}\,.\end{aligned}$$

We are allowed to divide by f since it never vanished.

By assumption f is differentiable at a, thus f is also continuous at a, and therefore

$$\lim_{h\to 0} f(a+h) = f(a) \neq 0\,.$$

In addition, since f is differentiable at a, by definition of derivative:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)\,.$$

Finally we use the limit laws:

$$\begin{aligned}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} &= -\frac{1}{f(a)}\cdot\frac{1}{\lim\limits_{h\to 0} f(a+h)}\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \\ &= -\frac{1}{f(a)}\cdot\frac{1}{f(a)}\cdot f'(a)\,,\end{aligned}$$

which is the statement I wanted to prove.

Question 3 *The power rule says that, for every $c \in \mathbb{R}$:*

$$\frac{d}{dx}\left[x^c\right] = cx^{c-1}$$

In this problem, we will restrict ourselves only to the domain $x > 0$.

In Video 3.7 you learned a proof for a particular case: when c is a positive integer. You will later (Video 4.10) learn a proof that works for all $c \in \mathbb{R}$ using logarithms, but there are other simple proofs, without using logarithms, that extend to $c \in \mathbb{Q}$. That is the goal of this problem.

You may assume the power rule when c is a positive integer, as well as other results you learned in Unit 3, including the Chain Rule.

(a) *Prove the power rule when c is a positive rational.*

Suggestion: *Assume $c = p/q$ for some positive integers p and q. Define the function $f(x) = x^{p/q}$. Then this function satisfies*

$$f(x)^q = x^p.$$

Use implicit differentiation.

(b) *Prove the power rule when c is a negative rational.*

Suggestion: *Look at what you have done so far in this assignment.*

Solution:

(a) Let c be a positive rational. It can be written as $c = p/q$, where p and q are positive integers. Define $f(x) = x^{p/q}$ for $x > 0$. My goal is to prove that $f'(x) = x^{c-1}$ using implicit differentiation.[2]

By definition, for every positive real number x, the value $f(x)$ is defined as the unique positive real number such that

$$(f(x))^q = x^p.$$

Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(f(x))^q\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^p\right]. \tag{1}$$

Using the Power Rule for positive-integer exponents (for exponent p), we can write the right-hand side in (1) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^p\right] = px^{p-1} \tag{2}$$

Using the Power Rule for positive-integer exponents (for exponent q) and the Chain Rule, we can write the left-hand side in (1) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)^q\right] = qf(x)^{q-1}f'(x) \tag{3}$$

[2]Strictly speaking, the calculation I will write below using implicit differentiation only allows us to obtain a formula for $f'(x)$ once we know that the function f is differentiable. You can try to prove this using the tools you learn in Unit 4. For now, we will ignore it.

Putting together (1), (2), and (3):

$$q f(x)^{q-1} f'(x) \;=\; p x^{p-1}.$$

From here we can solve for $f'(x)$ (since all the quantities are non-zero):

$$f'(x) \;=\; \frac{p\, x^{p-1}}{q\, f(x)^{q-1}}$$

Finally, we use the definition of f and the properties of powers:

$$f'(x) \;=\; \frac{p\, x^{p-1}}{q\, \left[x^{p/q}\right]^{q-1}} \;=\; \frac{p}{q}\, x^{(p-1)-\frac{p}{q}(q-1)} \tag{4}$$

The exponent of x on the right-hand side can be written as

$$(p-1) - \frac{p}{q}(q-1) \;=\; \frac{q(p-1) - p(q-1)}{q} \;=\; \frac{p-q}{q} \;=\; \frac{p}{q} - 1 \;=\; c - 1$$

And thus (4) becomes

$$f'(c) \;=\; c x^{c-1},$$

which is what I wanted to prove.

(b) Let c be a negative rational. My goal is to prove that $\dfrac{\mathrm{d}x^c}{\mathrm{d}x} = c x^{c-1}$.

Let us call $f(x) = x^{-c}$ and $g(x) = x^c$. Using the properties of powers (and since $x \neq 0$) we can write

$$g(x) \;=\; x^c \;=\; \frac{1}{x^{-c}} \;=\; \frac{1}{f(x)} \tag{5}$$

Since c is a negative rational, $-c$ must be a positive rational. We have already proven the Chain Rule for positive rational exponents in Question 3a. Therefore:

$$f'(x) \;=\; \frac{\mathrm{d}x^{-c}}{\mathrm{d}x} = -c x^{-c-1}\,. \tag{6}$$

Using the result in Question 2, we can write

$$g'(x) \;=\; \frac{-f'(x)}{\left[f(x)\right]^2} \tag{7}$$

Finally, putting together (6) and (7) we get

$$g'(x) \;=\; \frac{-f'(x)}{\left[f(x)\right]^2} \;=\; \frac{-\left(-c x^{-c-1}\right)}{\left(x^{-c}\right)^2} \;=\; c\, x^{-c-1+2c} \;=\; c\, x^{c-1}.$$

In the second-to-last step I have used the properties of powers. This is what I wanted to prove, so it concludes the proof.

Question 4 *The function h satisfies the following equation:*

$$\forall x \in \mathbb{R}, \quad h(xh(x)) = \left[h(x)\right]^3 . \tag{8}$$

In addition, we know:

- *The domain of h is $\mathbb{R}$.*
- *h is twice differentiable (meaning that h is differentiable, and h' is also differentiable).*
- *$h(1) = 1$.*
- *The graph of h does not have a horizontal tangent line at the point with x-coordinate 1.*

Calculate $h''(1)$.
Hint: *Use implicit differentiation.*

Solution:

Step 1: Obtain a relation between h and h'.

I will manipulate Equation (8) to obtain relations involving h and h'. Since h is differentiable, the product $xh(x)$ is also differentiable, and so is the composition $h(xh(x))$. Using the product rule I have

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(xh(x)\big) \;=\; h(x) + xh'(x) \tag{9}$$

while by the chain rule I have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(h\left(xh(x)\right)\Big) = h'\left(xh(x)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(xh(x)\right) = h'\left(xh(x)\right) \cdot \left(h(x) + xh'(x)\right) . \tag{10}$$

On the other hand, the function $h(x)^3$ is also differentiable, being obtained by composition of h and $g(y) = y^3$. The chain rule and the power rule then yield

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)^3\right) = 3h'(x)h(x)^2 . \tag{11}$$

I can now equate the derivatives of the two sides of Equation (8) to obtain

$$\boxed{h'\left(xh(x)\right) \cdot \left(h(x) + xh'(x)\right) = 3h'(x)h(x)^2} \tag{12}$$

which expresses a relation between h and h' as anticipated.

Step 2: Obtain a relation between h, h', and h''.

Now I want to repeat the process to obtain a relation involving h, h', and h''. Given that h and h' are both differentiable, the right-hand side of (12) is obtained from differentiable functions via operations of sum, product, and composition, and it is therefore differentiable itself. As a first step, I can use the chain rule and (9) once again to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(h'\left(xh(x)\right)\Big) = h''\left(xh(x)\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(xh(x)\right) = h''\left(xh(x)\right)\cdot\left(h(x)+xh'(x)\right)\,, \tag{13}$$

while by the sum and product rules I have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)+xh'(x)\right) = h'(x) + \left(h'(x)+xh''(x)\right) = 2h'(x)+xh''(x)\,. \tag{14}$$

Combining the product rule with the two relations above and (10), I can find

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(h(xh(x))\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(h'\left(xh(x)\right)\Big)\cdot\left(h(x)+xh'(x)\right)\\ &\quad + h'\left(xh(x)\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)+xh'(x)\right) =\\ &= h''(xh(x))\left(h(x)+xh'(x)\right)^2 + h'(xh(x))\left(2h'(x)+xh''(x)\right)\,,\end{aligned} \tag{15}$$

which gives the second derivative of the left-hand side of (8). Similarly, the product and power rules applied to the right-hand side of (11) show that $h(x)^3$ is also twice differentiable, and that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(h(x)^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3h'(x)h(x)^2\right) = 3h''(x)h(x)^2 + 6h'(x)^2h(x)\,. \tag{16}$$

Putting (15) and (16) together, I obtain the desired relation to be

$$\boxed{\begin{aligned}h''(xh(x))\left(h(x)+xh'(x)\right)^2 + h'(xh(x))\left(2h'(x)+xh''(x)\right) &=\\ = 3h''(x)h(x)^2 + 6h'(x)^2h(x)\,.&\end{aligned} \tag{17}}$$

Step 3: Evaluate at $x=1$ to obtain $h''(1)$.

The relations above hold for every $x\in\mathbb{R}$, and I can apply them in particular for $x=1$. Using that $h(1)=1$, (12) reads

$$h'(1)\cdot(1+h'(1)) = 3h'(1). \tag{18}$$

If $h'(1)=0$, then the graph of h would have a horizontal tangent line at $(1,1)$, which is excluded by the hypotheses: I can then conclude that $h'(1)\neq 0$ and divide both sides by $h'(1)$ to find

$$1+h'(1) = 3 \tag{19}$$

or $\boxed{h'(1) = 2}$. I can now substitute this and $h(1) = 1$ into (17) to obtain

$$\begin{aligned}
&h''(1) \cdot (1+2)^2 + 2\left(2 \cdot 2 + h''(1)\right) = 3h''(1) + 6 \cdot 2^2 \\
&9h''(1) + 8 + 2h''(1) = 3h''(1) + 24 \\
&8h''(1) = 16 \\
&\boxed{h''(1) = 2}.
\end{aligned} \tag{20}$$