MAT 137Y: Calculus with proofs
Assignment 7 - Sample Solutions

1. For every natural number n, we define the function F_n by the equation

$$F_n(x) = \int_0^x t^n e^t dt. \tag{1}$$

   (a) Use integration by parts to write F_n in terms of F_{n-1} for $n \geq 1$.

Solution: Let us fix $n \geq 1$. I use integration by parts

$$\int u\,dv = uv - \int v\,du$$

with choices

$$u = t^n \qquad du = nt^{n-1}$$
$$dv = e^t dt \qquad v = e^t$$

to get

$$\int t^n e^t dt = t^n e^t - n\int t^{n-1}e^t dt \tag{2}$$

Equation (2) is for an indefinite integral. Now let's make it a definite integral: from 0 to x:

$$\int_0^x t^n e^t dt = t^n e^t\Big|_{t=0}^{t=x} - n\int_0^x t^{n-1}e^t dt$$

Equivalently, using the definition of F_n:

$$\boxed{F_n(x) = x^n e^x - nF_{n-1}(x)} \tag{3}$$

(b) Prove the following theorem by induction, using your result from Question 1a:

Theorem. For every natural number n there exists a polynomial P_n and a real number λ_n such that

$$\forall x \in \mathbb{R}, \quad F_n(x) \;=\; e^x P_n(x) \;+\; \lambda_n$$

Solution: I will prove the theorem by induction on n.

- Base case ($n = 0$).

  I need to prove that there exists a polynomial P_0 and a real number λ_0 such that, for every $x \in \mathbb{R}$:

$$F_0(x) = e^x P_0(x) + \lambda_0. \tag{4}$$

  I can do this by direct calculation:

$$F_0(x) \;=\; \int_0^x e^t dt = \quad e^t \Big|_{t=0}^{t=x} \;=\; e^x - 1$$

  I have shown that Equation (4) is true for the constant polynomial $P_0(x) = 1$ and the real number $\lambda_0 = -1$.

- Induction step.

  Let us fix $n \geq 1$.
  I assume there exists a polynomial P_{n-1} and a real number λ_{n-1} such that, for every $x \in \mathbb{R}$:

$$F_{n-1}(x) \;=\; e^x P_{n-1}(x) \;+\; \lambda_{n-1}. \tag{5}$$

  I want to prove that there exists a polynomial P_n and a real number λ_n such that, for every $x \in \mathbb{R}$:

$$F_n(x) \;=\; e^x P_n(x) \;+\; \lambda_n. \tag{6}$$

  I can use the relation I obtained in the previous question, Equation (3), together with the induction hypothesis, Equation (5), to conclude that for every $x \in \mathbb{R}$:

$$\begin{aligned} F_n(x) \;&=\; x^n e^x \;-\; nF_{n-1}(x) \\ &=\; x^n e^x \;-\; n\left[e^x P_{n-1}(x) + \lambda_{n-1}\right] \\ &=\; e^x\left[x^n - nP_{n-1}(x)\right] \;-\; n\lambda_{n-1} \end{aligned}$$

  Notice that $x^n - nP_{n-1}(x)$ is a polynomial in x (because it is a sum of polynomials) and $-n\lambda_{n-1}$ is a real number. In other words, I have shown that Equation (6) is true for the polynomial

$$P_n(x) = x^n - nP_{n-1}(x)$$

  and for the real number

$$\lambda_n = -n\lambda_{n-1}.$$

(c) Find (and prove) an explicit formula for λ_n, as defined in Question 1b.
Hint: First find λ_0 by direct calculation. Then use the previous questions.

Solution:
In the previous question I concluded that

$$\lambda_0 = -1 \tag{7}$$
$$\forall n \geq 1, \quad \lambda_n = -n\lambda_{n-1} \tag{8}$$

As a consequence,

$$\boxed{\forall n \geq 0, \quad \lambda_n = (-1)^{n+1} n!} \tag{9}$$

I will prove (9) by induction on n.

- The base case ($n = 0$) is already proven by (7):

$$\lambda_0 = -1 = (-1)^{0+1} 0!$$

- For the induction step, let us fix $n \geq 1$. I assume that

$$\lambda_{n-1} = (-1)^n (n-1)! \tag{10}$$

  I need to prove that
$$\lambda_n = (-1)^{n+1} n!$$

  This follows immediately from (8) and the induction hypothesis, (10):

$$\lambda_n \;=\; -n\lambda_{n-1} \;=\; -n(-1)^n (n-1)! \;=\; (-1)^{n+1} n(n-1)! \;=\; (-1)^{n+1} n!$$

2. Use substitutions to write the following integrals in terms of the functions F_n (as defined by Equation (1)):

(a) $\displaystyle\int_1^x t^p\, e^{at}\, dt$

(b) $\displaystyle\int x^{2p+1}\, e^{-x^2}\, dx$

(c) $\displaystyle\int_1^x t^p\, (\ln t)^q\, dt,\quad$ for $x > 0$

(d) $\displaystyle\int (\sin^p x)\left(\cos^3 x\right) e^{\sin x}\, dx$

where $p, q \in \mathbb{N}$, $a \in \mathbb{R}$, $a \neq 0$.

Solution:

(a) I use the substitution $u = at$ with $du = adt$. Notice that $a \neq 0$, so I can divide by a.

$$\int_1^x t^p e^{at}\, dt \;=\; \int_a^{ax} \left(\frac{u}{a}\right)^p e^u \frac{1}{a} du \;=\; \frac{1}{a^{p+1}} \int_a^{ax} u^p e^u du \;=\; \boxed{\frac{F_p(ax) - F_p(a)}{a^{p+1}}}$$

(b) I use the substitution $u = -x^2$ with $du = -2xdx$. Then

$$\int x^{2p+1}\, e^{-x^2}\, dx \;=\; \int x^{2p}\, e^{-x^2}\, (xdx) \;=\; \int (-u)^p\, e^u\, \frac{-du}{2} \;=\; \frac{(-1)^{p+1}}{2} \int u^p e^u\, du$$
$$= \frac{(-1)^{p+1}}{2} F_p(u) + C \;=\; \boxed{\frac{(-1)^{p+1}}{2} F_p(-x^2) + C}$$

(c) First, I use the substitution $u = \ln t$ with $du = \dfrac{dt}{t}$. Therefore, $t = e^u$

$$\int_1^x t^p\, (\ln t)^q\, dt \;=\; \int_1^x t^{p+1}\, (\ln t)^q\, \frac{dt}{t} \;=\; \int_0^{\ln x} e^{(p+1)u}\, u^q\, du \;=\; \ldots$$

Next, I use the substitution $y = (p+1)u$ with $dy = (p+1)du$. Notice that $p \in \mathbb{N}$ so $p + 1 \neq 0$ and I can divide by $(p+1)$.

$$\ldots \;=\; \int_0^{(p+1)\ln x} e^y \left(\frac{y}{p+1}\right)^q \frac{dy}{p+1} \;=\; \frac{1}{(p+1)^{q+1}} \int_0^{(p+1)\ln x} y^q e^y\, dy$$
$$= \boxed{\frac{1}{(p+1)^{q+1}} F_q((p+1)\ln x)}$$

(d) I use the substitution $u = \sin x$ with $du = \cos x$. I also use the identity $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}
\int (\sin^p x)\left(\cos^3 x\right) e^{\sin x}\, dx &= \int (\sin^p x)\left(\cos^2 x\right) e^{\sin x}\, (\cos x\, dx) \\
&= \int (\sin^p x)\left(1 - \sin^2 x\right) e^{\sin x}\, (\cos x\, dx) \\
&= \int u^p(1 - u^2)e^u\, du \;=\; \int u^p e^u\, du \;-\; \int u^{p+2} e^u\, du \\
&= F_p(u) - F_{p+2}(u) + C \;=\; \boxed{F_p(\sin x) - F_{p+2}(\sin x) + C}
\end{aligned}$$

3. Before you attempt this question, work on the Practice Problems for Unit 10 (specifically the sections on Mass Density and Center of Mass). Otherwise the question won't make sense.

   Every time we have two point masses in a closed space, they generate something called "macguffin". If we have a mass m_1 at point P_1 and a mass m_2 at point P_2, then they generate a macguffin with value G given by

$$G = m_1 m_2 z^2$$

   where z is the distance between P and Q.

   If we have more than two masses, every pair of masses generates a macguffin. For example, if we have three masses (call them 1, 2, and 3) at three different points, then the total macguffin generated by them is the sum of

   - the macguffin generated by masses 1 and 2,
   - the macguffin generated by masses 1 and 3,
   - the macguffin generated by masses 2 and 3.

   (a) Assume we have N masses on N different positions on the x-axis: a mass m_1 at x_1, a mass m_2 at x_2, ..., a mass m_N at x_N. Obtain a formula for the total macguffin generated by the masses using sigma notation.

   (b) Assume that instead of a collection of point masses we have continuous masses (which is more realistic). Specifically, we have a bar on the x-axis, from $x = a$ to $x = b$, whose mass density at the point x is given by $\mu(x)$. Assume μ is a continuous function. Obtain a formula for the total macguffin generated by the bar using integrals.

Solution:

(a) The macguffin created by masses m_i and m_j will be

$$G_{i,j} = m_i m_j (x_i - x_j)^2$$

I have to consider all possible pairs of masses. I have to consider all possible values of i and j. This means I have to take a double sum. I may be tempted to write

$$G = \sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j (x_i - x_j)^2 \quad ??? \tag{11}$$

but this equation is wrong. There are two issues. First, in this double sum I am including the macguffin generated by a mass with itself, which I shouldn't! On closer inspection, this turns out not to be a problem because a term like

$$m_i m_i (x_i - x_i)^2$$

is always 0. So, it does not matter that I included this term in the sum. However, there is a second problem that needs to be fixed. In the double sum (11), I am including the macguffin generated by each pair of different masses twice. For example, I am including the macguffin generated by masses m_1 and m_2 once when $i = 1$ and $j = 2$, and a second time when $i = 2$ and $j = 1$. There are various ways to fix this. I can simply divide the full sum by two

$$\boxed{G = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j (x_i - x_j)^2} \tag{12}$$

Alternatively, to make sure I include each pair of masses only once, I can add, for each mass m_i only the contribution from masses m_j with $j \geq i$:

$$\boxed{G = \sum_{i=1}^{N} \sum_{j=i}^{N} m_i m_j (x_i - x_j)^2} \tag{13}$$

or with $j \leq i$:

$$\boxed{G = \sum_{i=1}^{N} \sum_{j=1}^{i} m_i m_j (x_i - x_j)^2} \tag{14}$$

And of course, perhaps I prefer to explicitly exclude the cases $j = i$:

$$\boxed{G = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} m_i m_j (x_i - x_j)^2} \tag{15}$$

or:

$$\boxed{G = \sum_{i=2}^{N} \sum_{j=1}^{i-1} m_i m_j (x_i - x_j)^2} \tag{16}$$

Any of the Equations (12), (13), (14), (15), (16) is a valid answer.

(b) For simplicity, I will focus on Equation (12) as the macguffin generated by a finite collection of masses. To obtain an equation for the macguffin generated by a continuous mass, I can take the Riemann sums approach, or the infinitesimals approach.

<u>Using Riemann sums</u>

Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of the interval $[a, b]$. For each i, I pick a point $x_i^\star$ on each subinterval $[x_{i-1}, x_i]$, and I define $\Delta x_i = x_i - x_{i-1}$. I can approximate the mass in the subinterval $[x_{i-1}, x_i]$ by a single point-mass with value $\mu(x_i^\star) \cdot \Delta x_i$ at the point $x_i^\star$. From (12), the macguffin generated by these point masses will be

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\mu(x_i^\star)\Delta x_i\right] \cdot \left[\mu(x_j^\star)\Delta x_j\right] \cdot (x_i^\star - x_j^\star)^2$$

This is an approximation. If we now take a limit as the norm of the partitions approaches 0, we will get an integral (well, a double integral). Notice that all the involved functions are continuous, and hence integrable.

$$\begin{aligned} G &= \lim_{||P||\to 0} \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} [\mu(x_i^\star)\Delta x_i] \cdot \left[\mu(x_j^\star)\Delta x_j\right] \cdot (x_i^\star - x_j^\star)^2 \\ &= \lim_{||P||\to 0} \frac{1}{2} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \left[\mu(x_i^\star)\mu(x_j^\star)(x_i^\star - x_j^\star)^2\right] \Delta x_j \right) \Delta x_i \\ &= \frac{1}{2} \int_a^b \left(\int_a^b \mu(x)\mu(y)(x-y)^2 dy \right) dx \end{aligned}$$

It is important to use two different variables of integration in the double integral. I used "dx" and "dy" but you could have used any other variables (as long as they are different).

To be fair, the last step is more an intuitive leap than a rigorous deduction. It can be made more rigorous (in particular, you will study iterated integrals in detail in MAT235, MAT237, or MAT257) but for now this will suffice.

Using infinitesimals

If I cut down an infinitesimally thin slice of the bar, at x, with width dx, then it has a mass of

$$\mu(x)dx$$

If I cut down a second infinitesimally thin slice of the bar, at y, with width dy, then it has a mass of

$$\mu(y)dy$$

and together they generate a macguffin with value

$$\mu(x)dx\, \mu(y)dy\, (x-y)^2$$

Now I need to add over all possible values of x and y. Since these are infinitesimal pieces, the "sum" is actually an integral (well, a double integral). Using the same arguments I used in Question 3a, I get

$$\boxed{G = \frac{1}{2} \int_a^b \left(\int_a^b \mu(x)\mu(y)(x-y)^2 dy \right) dx}$$

Alternative answers

If I use (13) or (15) instead of (12) I get

$$\boxed{G = \int_a^b \left(\int_x^b \mu(x)\mu(y)(x-y)^2 dy \right) dx}$$

And if I use (14) or (16) I get

$$\boxed{G = \int_a^b \left(\int_a^x \mu(x)\mu(y)(x-y)^2 dy \right) dx}$$

which are also correct. Notice that once I am down to integrals, the problem of the contribution of the macguffin generated by a single infinitesimal mass with itself ($x = y$) becomes even more irrelevant: an integral is unaffected if we change the value of a function at a single point.