MAT 137Y: Calculus with proofs
Assignment 8 - Sample Solutions

Notation: We will denote the set of positive integers by $\mathbb{Z}^+$.

1. Prove the following lemma:

   Lemma A. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We define two new sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ as:

   $$\forall n \in \mathbb{Z}^+, \ E_n = x_{2n},$$
   $$\forall n \in \mathbb{Z}^+, \ O_n = x_{2n-1}$$

   - IF the sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ are both convergent to the ***same*** limit,
   - THEN the sequence $\{x_n\}_{n=1}^{\infty}$ is also convergent.

   Suggestion: Use the definition of limit.

   Solution:

   - Let me call $L = \lim_{n\to\infty} E_n = \lim_{n\to\infty} O_n$. I can write this because we are assuming the two sequences are convergent to the same limit.
   - I am going to show that $L = \lim_{n\to\infty} x_n$. In other words, I want to show that

   $$\forall \varepsilon > 0, \exists n_0 \in \mathbb{Z}^+, \forall n \in \mathbb{Z}^+, \quad n \geq n_0 \implies |x_n - L| < \varepsilon. \tag{1}$$

   - $\boxed{\text{Let } \varepsilon > 0.}$
     - From the definition of $\lim_{n\to\infty} O_n = L$ (using this same ε),
       I conclude $\exists n_1 \in \mathbb{Z}^+$ such that

       $$\forall n \in \mathbb{Z}^+, \quad n \geq n_1 \implies |O_n - L| < \varepsilon \tag{2}$$

     - From the definition of $\lim_{n\to\infty} E_n = L$ (using this same ε),
       I conclude $\exists n_2 \in \mathbb{Z}^+$ such that

       $$\forall n \in \mathbb{Z}^+, \quad n \geq n_2 \implies |E_n - L| < \varepsilon \tag{3}$$

     $\boxed{\text{I take } n_0 = \max\{2n_1 - 1, 2n_2\}.}$ I will show this value of n_0 works as in (1).
   - $\boxed{\text{Let } n \in \mathbb{Z}^+. \text{ Assume } n \geq n_0}$. I need to show that $|x_n - L| < \varepsilon$.
   - I will break into two cases, depending on whether n is odd or even.

- *Case 1: n is odd.*
  In this case, $n = 2k - 1$ for some $k \in \mathbb{Z}^+$ and $x_n = x_{2k-1} = O_k$. Then

$$2k - 1 \; = \; n \; \geq \; n_0 \; \geq \; 2n_1 - 1$$

  Therefore $2k - 1 \geq 2n_1 - 1$ and hence $k \geq n_1$. It then follows from (2) that

$$|x_n - L| \; = \; |O_k - L| \; < \; \varepsilon$$

- *Case 2: n is even.*
  In this case, $n = 2k$ for some $k \in \mathbb{Z}^+$ and $x_n = x_{2k} = E_k$. Then

$$2k \; = \; n \; \geq \; n_0 \; \geq \; 2n_2$$

  Therefore $2k \geq 2n_2$ and hence $k \geq n_2$. It then follows from (3) that

$$|x_n - L| \; = \; |E_k - L| \; < \; \varepsilon$$

In both cases $\boxed{\text{I have proven that } |x_n - L| < \varepsilon.}$

$\square$

2. Let $\{a_n\}_{n=1}^{\infty}$ be a *decreasing* sequence of positive numbers with limit 0. I define a new sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$\begin{aligned} x_1 &= a_1 \\ \forall n \in \mathbb{Z}^+, \quad x_{n+1} &= x_n + (-1)^n a_{n+1} \end{aligned} \tag{4}$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the hypotheses of Lemma A, and hence is it convergent.

This question is quite long and you will need to prove a few different things. Before you start, make a strategy. Decide what the various things you need to prove are, and in which order. Begin the proof by writing a summary of the steps you are going to take. Make sure your reader understands where in your proof you are at each moment. Make your proof as easy to read as you would like it to be if you were reading it for the first time yourself.

Suggestions: You do not need to write a single ε! Use the theorems you have learned in Unit 11 instead. Before you start, as rough work, write explicitly an equation for the first few x's in terms of the first few a's to make sure you understand the definition. Draw the numbers x_1, x_2, x_3, x_4, x_5, x_6, x_7, and x_8 in a real line and order them. You will notice that the sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ appear to satisfy certain properties which will be helpful in your proof. Of course, anything you want to use in your proof needs to be proven first.

Solution:

Before starting notice that the recurrence relation in (4) means that, for every $n \in \mathbb{Z}^+$:

$$x_{2n} = x_{2n-1} - a_{2n} \tag{5}$$

$$x_{2n+1} = x_{2n} + a_{2n+1} \tag{6}$$

I define the sequences $\{O_n\}_{n=1}^{\infty}$ and $\{E_n\}_{n=1}^{\infty}$ as in Lemma A. I need to prove that these two sequences are both convergent and have the same limit. Before I begin, looking at the first few terms and doing some algebra, it looks like the terms are ordered like this:

$$x_2 < x_4 < x_6 < x_8 < \ldots < x_7 < x_5 < x_3 < x_1$$

This suggests the following plan:

- *Claim 1:* The sequence $\{E_n\}_{n=1}^{\infty}$ is increasing.
- *Claim 2:* The sequence $\{O_n\}_{n=1}^{\infty}$ is decreasing.
- *Claim 3:* For every $n \in \mathbb{Z}^+$, $\quad E_n < O_n$
- *Claim 4:* The sequence $\{E_n\}_{n=1}^{\infty}$ is bounded above (by O_1).
- *Claim 5:* The sequence $\{O_n\}_{n=1}^{\infty}$ is bounded below (by E_1).
- *Claim 6:* The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ are both convergent.
- *Claim 7:* The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ have the same limit.

Note: The order of these claims matters. Specifically:

- I will use Claims 2 and 3 in the proof of Claim 4.
- I will use Claims 1 and 3 in the proof of Claim 5.
- I will use Claims 1, 2, 4, and 5 in the proof of Claim 6.
- I will use Claim 6 in the proof of Claim 7.

Of course, you may have a different plan for the proof, but make sure your plan makes sense chronologically.

Claim 1: The sequence $\{E_n\}_{n=1}^{\infty}$ is increasing

Proof: Let $n \in \mathbb{Z}^+$. I need to prove that $E_{n+1} - E_n > 0$.

$$\begin{aligned} E_{n+1} = x_{2n+2} &= x_{2n+1} - a_{2n+2} && \text{(using (5))} \\ &= [x_{2n} + a_{2n+1}] - a_{2n+2} && \text{(using (6))} \\ &= E_n + a_{2n+1} - a_{2n+2} \end{aligned}$$

Therefore

$$E_{n+1} - E_n = a_{2n+1} - a_{2n+2} > 0 \qquad \text{(because the sequence } \{a_k\}_{k=1}^{\infty} \text{ is decreasing)}$$

□

Claim 2: The sequence $\{O_n\}_{n=1}^{\infty}$ is decreasing

Proof:

Let $n \in \mathbb{Z}^+$. I need to prove that $O_{n+1} - O_n < 0$.

$$\begin{aligned} O_{n+1} = x_{2n+1} &= x_{2n} + a_{2n+1} && \text{(using (6))} \\ &= [x_{2n-1} - a_{2n}] + a_{2n+1} && \text{(using (5))} \\ &= O_n - a_{2n} + a_{2n+1} \end{aligned}$$

Therefore

$$O_{n+1} - O_n = a_{2n+1} - a_{2n} < 0 \qquad \text{(because the sequence } \{a_k\}_{k=1}^{\infty} \text{ is decreasing)}$$

□

Claim 3: For every $n \in \mathbb{Z}^+$, $\quad E_n < O_n$

Proof:

Let $n \in \mathbb{Z}^+$. I want to prove that $O_n - E_n > 0$. Using (5):

$$E_n \;=\; x_{2n} \;=\; x_{2n-1} - a_n \;=\; O_n - a_n,$$

so that $O_n - E_n = a_n$, which we know is positive. ☐

Claim 4: The sequence $\{E_n\}_{n=1}^{\infty}$ is bounded above (by O_1).

Proof: Let $n \in \mathbb{Z}^+$. Then

- $E_n < O_n$ from Claim 3
- $O_n \leq O_1$ from Claim 2

I have proven that for every $n \in \mathbb{Z}^+$, $E_n \leq O_1$. ☐

Claim 5: The sequence $\{O_n\}_{n=1}^{\infty}$ is bounded below (by E_1).

Proof: Let $n \in \mathbb{Z}^+$. Then

- $O_n > E_n$ from Claim 3
- $E_n \geq E_1$ from Claim 1

I have proven that for every $n \in \mathbb{Z}^+$, $O_n \geq E_1$. ☐

Claim 6: The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ are both convergent.

Proof

- The sequence $\{E_n\}_{n=1}^{\infty}$ is increasing (Claim 1) and bounded above (Claim 4). By the Monotone Convergence Theorem, it is convergent.
- The sequence $\{O_n\}_{n=1}^{\infty}$ is decreasing (Claim 2) and bounded below (Claim 5). By the Monotone Convergence Theorem, it is convergent. ☐

Claim 7: The sequences $\{E_n\}_{n=1}^{\infty}$ and $\{O_n\}_{n=1}^{\infty}$ have the same limit.

Let us call

$$L_1 \;=\; \lim_{n\to\infty} E_n, \qquad L_2 \;=\; \lim_{n\to\infty} O_n$$

I can do this because I know both sequences converge (Claim 6.) I need to prove that $L_1 = L_2$. From (5) we know that for every $n \in \mathbb{Z}^+$:

$$x_{2n} \;=\; x_{2n-1} \;-\; a_{2n}$$

Or, in other words:

$$E_n \;=\; O_n \;-\; a_{2n}$$

Since all the terms have a limit, we can use a limit law:

$$\lim_{n\to\infty} E_n \;=\; \left[\lim_{n\to\infty} O_n\right] \;-\; \left[\lim_{n\to\infty} a_{2n}\right]$$

and finally

$$L_1 \;=\; L_2 \;+\; 0$$

because we knew the sequence $\{a_k\}_{k=1}^{\infty}$ was convergent to 0. I have proven that $L_1 = L_2$ as needed. $\square$

3. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of positive numbers. Assume that $x_n << y_n$. For each one of the following claims, decide whether they are always true, always false, or sometimes true and sometimes false (depending on the specific sequences). Prove it.

(a) $x_n << \dfrac{x_n + y_n}{2}$

(b) $\dfrac{x_n + y_n}{2} << y_n$

Solution:

(a) This is ALWAYS TRUE.

By definition of $x_n << y_n$, we know that $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = 0$. Then

$$\lim_{n\to\infty} \frac{x_n}{\frac{x_n + y_n}{2}} = \lim_{n\to\infty} \frac{2x_n}{x_n + y_n} = \lim_{n\to\infty} \frac{2\frac{x_n}{y_n}}{\frac{x_n}{y_n} + 1} = \frac{2 \cdot 0}{0 + 1} = 0$$

Therefore, by definition, $x_n << \dfrac{x_n + y_n}{2}$.

(b) This is ALWAYS FALSE.

By definition of $x_n << y_n$, we know that $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = 0$.

The definition of $\dfrac{x_n + y_n}{2} << y_n$ is

$$\lim_{n\to\infty} \frac{\frac{x_n + y_n}{2}}{y_n} = 0,$$

However, this limit is never 0:

$$\lim_{n\to\infty} \frac{\frac{x_n + y_n}{2}}{y_n} = \lim_{n\to\infty} \frac{x_n + y_n}{2y_n} = \lim_{n\to\infty} \frac{\frac{x_n}{y_n} + 1}{2} = \frac{0 + 1}{2} = \frac{1}{2}$$