MAT 137Y: Calculus with proofs
Assignment 9 - Sample Solutions

1. (a) Prove the following Theorem

Theorem 1. Let $\sum_{n=1}^{\infty} a_n$ be a series.

IF $\begin{cases} \lim_{k\to\infty} \sum_{n=1}^{2k} a_n & \text{exists} \\ \lim_{n\to\infty} a_n = 0 \end{cases}$

THEN the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Hint: You may use results from past assignments. They will make things simpler.

Solution:

Let us call $S_k = \sum_{n=1}^{k} a_n$.

As you know, we define the series as the limit of its partial sums

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} S_k$$

To prove the series $\sum_{n=1}^{\infty} a_n$ is convergent, I just need to prove that the sequence $\{S_k\}_{k=1}^{\infty}$ is convergent. Instead, I am going to prove that the two sequences $\{S_{2k}\}_{k=1}^{\infty}$ and $\{S_{2k-1}\}_{k=1}^{\infty}$ are convergent to the *same* limit. Then, using Lemma A from A8, I can conclude that the sequence $\{S_k\}_{k=1}^{\infty}$ is convergent.

- The first hypothesis means that the sequence $\{S_{2k}\}_{k=1}^{\infty}$ is convergent.
- Notice that, for every $k \geq 1$:

$$S_{2k-1} \;=\; S_{2k} - a_{2k}$$

- The first hypothesis means that

$$\lim_{k\to\infty} S_{2k} \;\; \text{exists}$$

The second hypothesis implies that

$$\lim_{k\to\infty} a_{2k} = 0$$

Therefore, I can use the limit law for sums and conclude that

$$\begin{aligned}
\lim_{k\to\infty} S_{2k-1} &= \lim_{k\to\infty} [S_{2k} - a_{2k}] \\
&= \left[\lim_{k\to\infty} S_{2k}\right] - \left[\lim_{k\to\infty} a_{2k}\right] \\
&= \left[\lim_{k\to\infty} S_{2k}\right] - [0] \\
&= \lim_{k\to\infty} S_{2k}
\end{aligned}$$

- I have proven that the two sequences $\{S_{2k}\}_{k=1}^{\infty}$ and $\{S_{2k-1}\}_{k=1}^{\infty}$converge to the same limit. Then, using Lemma A from A8, I conclude that the sequence $\{S_k\}_{k=1}^{\infty}$ is convergent, which is what I wanted to prove.

(b) Prove that the theorem is false if we remove any one of the two hypotheses.

Solution:

I need to provide

- An example that satisfies the first hypothesis but fails the conclusion.
- An example that satisfies the second hypothesis but fails the conclusion.

Notice that it is not necessary to prove that each example fails the other hypothesis. That is guaranteed (since otherwise it would not fail the conclusion.)

- **Example 1:** Consider the series $\sum_{n=1}^{\infty}(-1)^n$.

  We know the series is divergent (geometric with ratio -1), so it fails the conclusion. However, for every $k \geq 1$, $\sum_{n=1}^{2k}(-1)^n = 0$, and hence $\lim_{k\to\infty}\sum_{n=1}^{2k}(-1)^n = 0$. Therefore, it satisfies the first hypothesis.

- **Example 2:** Consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$.

  We know the series is divergent (p-series with $p = 1$), so it fails the conclusion. However, $\lim_{n\to\infty}\frac{1}{n} = 0$. Therefore, it satisfies the second hypothesis.

2. Let $\sum_{n}^{\infty} a_n$ be a CONVERGENT, NON-NEGATIVE series. Let f be a continuous function with domain $\mathbb{R}$. Decide whether each of the following series must be convergent, must be divergent, or we do not have enough information to decide. Prove it.

(a) $\sum_{n}^{\infty} (n^n \cdot a_n)$

Solution: NOT ENOUGH INFORMATION

We can prove this with two examples.

- Example 1: take $a_n = \frac{1}{n^2}$.
  - We know $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} \frac{1}{n^2}$ is convergent. (p-series with $p = 2$)
  - In this case $\sum_{n}^{\infty} (n^n \cdot a_n) = \sum_{n}^{\infty} n^{n-2}$ is divergent.

    This is by the necessary condition for convergence (Video 13.8), as $\lim_{n\to\infty} n^{n-2} = \infty \neq 0$.

- Example 2: take $a_n = \frac{1}{n^{n+2}}$.
  - We know $\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} \frac{1}{n^{n+2}}$ is convergent. (BCT: $0 \leq \frac{1}{n^{n+2}} \leq \frac{1}{n^2}$)
  - In this case $\sum_{n}^{\infty} (n^n \cdot a_n) = \sum_{n}^{\infty} \frac{1}{n^2}$ is convergent. (see above)

(b) $\displaystyle\sum_{n}^{\infty} \ln(2+a_n)$

Solution: DIVERGENT.

- Since $\displaystyle\sum_{n}^{\infty} a_n$ is convergent, we know $\displaystyle\lim_{n\to\infty} a_n = 0$.
  This is by the necessary condition for convergence (Video 13.8).

- Therefore, since $f(x) = \ln(2+x)$ is continuous at 0: $\displaystyle\lim_{n\to\infty} \ln(2+a_n) = \ln 2$.

- Since $\ln 2 \neq 0$, this means that $\displaystyle\sum_{n}^{\infty} \ln(2+a_n)$ is divergent.
  Again, this is by the necessary condition for convergence (Video 13.8).

(c) $\sum_{n}^{\infty} \ln \dfrac{2+a_{n+1}}{2+a_n}$

Solution: CONVERGENT

- This series is telescopic, so we can add it up. We know that, for the purpose of convergence, it does not matter where the series begins (as long as all the terms are defined). Let's assume it begins at $n = 1$ and let's give a name to the partial sums. (If this makes you uncomfortable, then call the starting index n_0 instead – the proof below will work the same.)

$$S_N = \sum_{n=1}^{N} \ln \frac{2+a_{n+1}}{2+a_n}$$

- Then we can write

$$\begin{aligned} S_N &= \ln \frac{2+a_2}{2+a_1} + \ln \frac{2+a_3}{2+a_2} + \ln \frac{2+a_4}{2+a_3} \ldots + \ln \frac{2+a_{N+1}}{2+a_N} \\ &= \ln \frac{(2+a_2)\cdot(2+a_3)\cdot(2+a_4)\cdot\ldots\cdot(2+a_{N+1})}{(2+a_1)\cdot(2+a_2)\cdot(2+a_3)\cdot\ldots\cdot(2+a_N)} \\ &= \ln \frac{2+a_{N+1}}{2+a_1} \end{aligned}$$

- By assumption $\sum_{n}^{\infty} a_n$ is convergent. Hence, by the necessary condition for convergence (Video 13.8), $\lim_{n\to\infty} a_n = 0$.
- Therefore

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln \frac{2+a_{N+1}}{2+a_1} = \ln \frac{2}{2+a_1}$$

where I have used that the function $g(x) = \ln \dfrac{2+x}{2+a_1}$ is continuous at $x = 0$.

- In particular, $\lim_{N\to\infty} S_N$ exists. This, by definition, means that the series $\sum_{n}^{\infty} \ln \dfrac{2+a_{n+1}}{2+a_n}$ is convergent.

(d) $\displaystyle\sum_{n}^{\infty} \ln(1+a_n)$

Solution: CONVERGENT

Method 1 (Using BCT)

- In order to use BCT, I first want to prove that, for all $x \geq 0$, $\ln(1+x) \leq x$.
- Consider the function g defined by $g(x) = x - \ln(1+x)$ with domain $(-1, \infty)$. In its domain
$$g'(x) \;=\; 1 - \frac{1}{1+x} \;=\; \frac{x}{1+x}$$
We see that for all $x > 0$, $g'(x) > 0$. Moreover, g is continuous on $[0, \infty)$. Therefore, g is increasing on $[0, \infty)$. We conclude that
$$\forall x > 0, \quad g(x) = x - \ln(1+x) \;>\; g(0) = 0$$
Hence, for all $x \geq 0$
$$\ln(1+x) \leq x.$$
- Now we can use BCT. Notice that for all n, $a_n \geq 0$ by assumption. Hence, for all n:
$$0 \;\leq\; \ln(1+a_n) \;\leq\; a_n.$$
Since we know $\displaystyle\sum_{n}^{\infty} a_n$ is convergent, using BCT we conclude that $\displaystyle\sum_{n}^{\infty} \ln(1+a_n)$ is also convergent.

Method 2 (Using LCT)

- By assumption $\displaystyle\sum_{n}^{\infty} a_n$ is convergent. Hence, by the necessary condition for convergence (Video 13.8), $\displaystyle\lim_{n\to\infty} a_n = 0$.
- Using L'Hôpital's Rule:
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{(L'H)}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} \;=\; 1$$
- I would like to say that then
$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} \;=\; 1$$
Using LCT, since I know $\displaystyle\sum_{n}^{\infty} a_n$ is convergent, I conclude that $\displaystyle\sum_{n}^{\infty} \ln(1+a_n)$ is also convergent. **However**, there is a flaw in this argument: LCT only works for strictly positive series (since otherwise I am dividing by 0). We can resolve this by breaking the problem into cases:

- Case 1: If $a_n > 0$ for all n, then the above argument is correct.

- Case 2: Let's assume $a_n = 0$ for some values of n, but there are still infinitely many values of n for which $a_n \geq 0$. In that case, we can first "remove all the zeroes" and then use LCT. More precisely, we can define a new sequence $\{b_k\}_{k=1}^{\infty}$ by

$$b_k \;=\; k\text{-th non-zero term in the sequence } \{a_n\}$$

  Then

  * $\displaystyle\sum_{k=1}^{\infty} b_k \;=\; \sum_{n}^{\infty} a_n,$ so it is convergent.
  * $\displaystyle\sum_{k=1}^{\infty} \ln(1+b_k)$ is convergent because we can use LCT on $\displaystyle\sum_{k=1}^{\infty} b_k$ and $\displaystyle\sum_{k=1}^{\infty} \ln(1+b_k)$.
  * $\displaystyle\sum_{n}^{\infty} \ln(1+a_n) \;=\; \sum_{k=1}^{\infty} \ln(1+b_k),$ so it is convergent.

- Case 3: If $a_n = 0$ except for finitely many values of n (possibly none), then a tail of the series $\displaystyle\sum_{n}^{\infty} a_n$ consists of only zeroes, and the result is trivial.

(e) $\sum_{n}^{\infty} (-1)^n \sqrt{a_n}$

Solution: NOT ENOUGH INFORMATION.[1]

We can prove this with two examples.

- Example 1: take $a_n = \dfrac{1}{n^2}$
  - We know $\displaystyle\sum_{n}^{\infty} a_n = \sum_{n}^{\infty} \frac{1}{n^2}$ is convergent (p-series with $p = 2$)
  - In this case $\displaystyle\sum_{n}^{\infty} (-1)^n \sqrt{a_n} = \sum_{n}^{\infty} \frac{(-1)^n}{n}$ is convergent (Alternating Series Test)

- Example 2: take $\{a_n\}_{n=1}^{\infty}$ as the sequence

$$0,\ 1,\ 0,\ \frac{1}{2^2},\ 0,\ \frac{1}{3^2},\ 0,\ \frac{1}{4^2},\ 0,\ \frac{1}{5^2},\ \ldots$$

  Or, equivalently

$$a_n = \begin{cases} 1/k^2 & \text{if } n = 2k \text{ for some } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}$$

  - We know $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent (p-series with $p = 2$)
  - In this case $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{a_n} = \sum_{k=1}^{\infty} \frac{1}{k}$ is divergent (p-series with $p = 1$)

    To justify this last step, notice that

$$\sum_{n=1}^{\infty} (-1)^n \sqrt{a_n} = 0 + 1 + 0 + \frac{1}{2} + 0 + \frac{1}{3} + 0 + \frac{1}{4} + \ldots$$

    which looks just like the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

    but with additional 0 interspaced. Intuitively the two infinite sums appear to be the same. We can also show this rigorously, but showing that the even partial

[1] If you thought that the answer was "convergent" due to the Alternating Series Test, notice that we cannot necessarily use that test: we do not know whether the sequence $\{a_n\}$ is (eventually) decreasing.

sums and the odd partial sums of $\sum_{n=1}^{\infty}(-1)^n\sqrt{a_n}$ are both equal to the partial sums of $\sum_{k=1}^{\infty}\frac{1}{k}$.

(f) $\sum_{n}^{\infty} (a_n\, f(\sin n))$

Solution: CONVERGENT

- We know f is a continuous function with domain $\mathbb{R}$. By EVT, it is bounded on $[-1,1]$. In other words, $\exists M > 0$ such that

$$\forall x \in [-1,1],\ |f(x)| \leq M$$

- Notice that for every n, $|\sin n| \in [-1,1]$. Therefore:

$$0 \ \leq\ |a_n f(\sin n)| \ =\ a_n |f(\sin n)| \ \leq\ M a_n \tag{1}$$

  where I have used that $a_n \geq 0$.

- We know that $\sum_{n}^{\infty} M a_n$ is convergent, because we know $\sum_{n}^{\infty} a_n$ is convergent and by linearity of series (Video 13.6).

  Thus, using BCT on (1) we conclude that $\sum_{n}^{\infty} |a_n f(\sin n)|$ is also convergent.

- Therefore, $\sum_{n}^{\infty} a_n f(\sin n)$ is convergent. (Absolute convergence test)