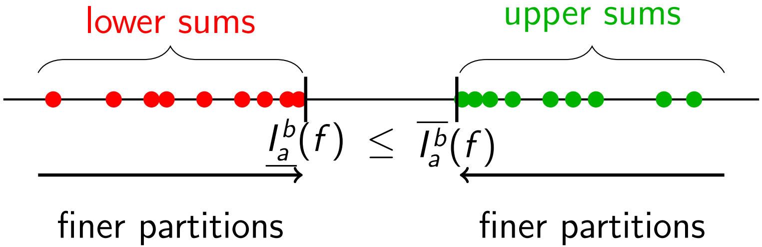
- Today: Examples and properties of the integral
- WEDNESDAY: Integrals as limits **(Videos 7.9, 7.10)**



Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \qquad \int_0^4 f(x)dx = 9, \qquad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(x)dx$
3. $\int_0^2 f(t)dx$
4. $\int_2^0 f(x)dx$
5. $\int_2^4 f(x)dx$
6. $\int_{-2}^0 f(x)dx$
7. $\int_0^4 [f(x) - 2g(x)]\ dx$

Example 1

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P with exactly 3 points (2 subintervals) such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

Finer partitions

Let f be a bounded function on $[a, b]$.
Let P and Q be partitions of $[a, b]$.
Which of these implications are true?

1. IF $P \subseteq Q$, THEN $L_P(f) \leq L_Q(f)$
2. IF $P \subset Q$, THEN $L_P(f) < L_Q(f)$
3. IF $L_P(f) \leq L_Q(f)$, THEN $P \subseteq Q$
4. IF $L_P(f) < L_Q(f)$, THEN $P \subset Q$
5. IF $L_P(f) < L_Q(f)$, THEN $Q \not\subseteq P$

An alternative definition

Recall

Let $A \subseteq \mathbb{R}$. The supremum of A as the only real number S such that...

- S is an upper bound of A.
- $\forall \varepsilon > 0, \quad \exists x \in A \quad \text{such that} \quad S - \varepsilon < x$.

Complete the following alternative definition of lower integral:

Let f be a bounded function on the interval $[a, b]$.
$\underline{I_a^b}(f)$ is the only real number that satisfies these two properties:

1. $\forall$ partition P of $[a, b]$, ...
2. $\forall \varepsilon > 0$, ...

Do the same thing for upper integral.