MAT 137Y: Calculus with proofs
Test 3 - Part B - Solutions

QUESTION 1

Q1 - The problem

$0 \cdot \infty$ is a limit indeterminate form. This means that, when computing a limit, if we run into the form $0 \cdot \infty$, we cannot conclude what the value of the limit will be based only on this information. Prove that the actual value of such a limit can be *any* real number.
You may use limits at a point (at a real number), or side limits, or limits at ∞ or at $-\infty$, whatever you prefer.

Q1 - Solution

This question is similar to Video 6.11, time 1:20-3:00. In that video we justified that 1^∞ was an indeterminate form. I will do the same thing for $0 \cdot \infty$.

Let $c \in \mathbb{R}$. I will construct an example of a limit indeterminate form of type $0 \cdot \infty$ with actual limit c.

- Let f be the function defined by $f(x) = x$. Notice that $\lim_{x \to \infty} f(x) = \infty$.
- Let g be the function defined by $g(x) = \frac{c}{x}$. Notice that $\lim_{x \to \infty} g(x) = 0$.
- Therefore, the limit of the product
$$\lim_{x \to \infty} g(x)f(x)$$
is an indeterminate form of type $0 \cdot \infty$. Moreover
$$\lim_{x \to \infty} g(x)f(x) \; = \; \lim_{x \to \infty} \left[\frac{c}{x} \cdot x \right] \; = \; \lim_{x \to \infty} c \; = \; c$$

Notice that c was an arbitrary real number. This proves that a limit indeterminate form of type $0 \cdot \infty$ can end up being *any* real number.

QUESTION 2

Q2 - The problem

Let $a < b$. Let f be a bounded function on $[a.b]$. As you know, we define the lower integral of f on $[a, b]$ as a supremum, specifically:

$$\underline{I_a^b}(f) \;=\; \sup\left\{L_P(f) \mid P \text{ is a partition of } [a,b]\right\}$$

This is well-defined. In other words, this set must necessarily have a supremum. Prove it.
Note: In the videos we skipped this detail. We just stated that the supremum must necessarily exist and we left it as an exercise for you to justify why.

Q2 - Solution

See Video 7.5, time 7:33-7:42.

Let us give a name to the set:

$$A = \left\{L_P(f) \mid P \text{ is a partition of } [a,b]\right\}.$$

I will prove that A is bounded above and non-empty. Then, by the least-upper-bound property, it must have a supremum.

- The set A is not empty because there exists at least one partition. For example, let us take the trivial partition $Q = \{a, b\}$. Then $L_Q(f) \in A$, the set A has at least one element, and so it is not empty.

- To show A is bounded above let us fix one single partition. Once again, for example, let us take trivial partition $Q = \{a, b\}$. Then, by the properties of lower and upper sums we know that

  $$\forall \text{ partitions } P \text{ of } [a,b],\; L_P(f) \leq U_Q(f)$$

  Therefore, every number in A is less than or equal to $U_Q(f)$. Therefore, $U_Q(f)$ is an upper bound of A. Therefore, A is bounded above.

Note: When proving that A is bounded above, it is important to bound *all* the lower sums with the *same* upper sum. You need to find one single number x such that $L_P(f) \leq x$ for all partitions P; the number x cannot depend on P. In particular, if you write that for all partitions P, $L_P(f) \leq U_P(f)$, you are not finding an upper bound for A.

QUESTION 3

Q3 - The problem

Let f be a continuous function with domain $\mathbb{R}$. Compute the limit

$$\lim_{x\to 0} \int_0^x \frac{(x-t)\, f(t)}{x^2}\, dt.$$

Q3 - Solution: Method 1

$\boxed{\text{This is somewhat similar to Q7 on Unit 8 practice problems.}}$

First, notice that for the purpose of the integral the variable is 't'; we can treat 'x' as a constant for the purpose of the integral and "factor it out". This allows us to rewrite the limit as:

$$L_1 \;=\; \lim_{x\to 0} \int_0^x \frac{(x-t)\, f(t)}{x^2}\, dt \;=\; \lim_{x\to 0} \frac{x \int_0^x f(t)\, dt \;-\; \int_0^x t f(t)\, dt}{x^2}$$

This is an indeterminate form of type $0/0$. We can try to use L'Hôpital's Rule. To take the derivative of the numerator we need to use the Product Rule and FTC. Then we will get the limit:

$$\begin{aligned}
L_2 \;&=\; \lim_{x\to 0} \frac{\int_0^x f(t)\, dt \;+\; x \frac{d}{dx}\left[\int_0^x f(t)\, dt\right] \;-\; \frac{d}{dx}\left[\int_0^x t f(t)\, dt\right]}{2x} \\
&=\; \lim_{x\to 0} \frac{\int_0^x f(t)\, dt \;+\; x f(x) \;-\; x f(x)}{2x} \\
&=\; \lim_{x\to 0} \frac{\int_0^x f(t)\, dt}{2x}
\end{aligned}$$

This is again an indeterminate form of type $0/0$. We can try to use L'Hôpital's Rule again. To take the derivative of the numerator we need to use FTC again. Then we will get the limit:

$$L_3 \;=\; \lim_{x\to 0} \frac{\frac{d}{dx}\left[\int_0^x f(t)\, dt\right]}{2} \;=\; \lim_{x\to 0} \frac{f(x)}{2} \;=\; \frac{f(0)}{2}$$

In the last step I have used that f is continuous.
So this limit exists. Therefore, both applications of L'Hôpital's Rule were legal and we get

$$L_1 = L_2 = L_3 = \frac{f(0)}{2}.$$

Q3 - Solution: Method 2

First, notice that for the purpose of the integral the variable is 't'; we can treat 'x' as a constant for the purpose of the integral and "factor it out". This allows us to rewrite the limit as:

$$L_1 \;=\; \lim_{x\to 0} \int_0^x \frac{(x-t)\, f(t)}{x^2}\, dt \;=\; \lim_{x\to 0} \left[\frac{\int_0^x f(t)\, dt}{x} \;-\; \frac{\int_0^x t f(t)\, dt}{x^2} \right] \tag{1}$$

The function whose limit we are computing is the difference of two functions. Let's see if each piece has a limit separately. First we look at

$$\lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x}$$

This is an indeterminate form of type 0/0. We can try to use L'Hôpital's Rule. To take the derivative of the numerator we need to use FTC:

$$\lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x} \overset{(L'H)}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left[\int_0^x f(t)\,dt\right]}{1} = \lim_{x\to 0} \frac{f(x)}{1} = f(0) \tag{2}$$

In the last step we have used that f is continuous. Since this limit exists, the use of L'Hôpital's Rule was justified and the calculation is correct.

Next, let's look at

$$\lim_{x\to 0} \frac{\int_0^x tf(t)\,dt}{x^2}$$

This is an indeterminate form of type 0/0. We can try to use L'Hôpital's Rule. To take the derivative of the numerator we need to use FTC:

$$\lim_{x\to 0} \frac{\int_0^x tf(t)\,dt}{x^2} \overset{(L'H)}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left[\int_0^x tf(t)\,dt\right]}{2x} = \lim_{x\to 0} \frac{xf(x)}{2x} = \lim_{x\to 0} \frac{f(x)}{2} = \frac{f(0)}{2} \tag{3}$$

In the last step we have used again that f is continuous. Since this limit exists, the use of L'Hôpital's Rule was justified and the calculation is correct.

We see the two pieces in (1) have limits independently. We can use the limit laws and plug (2) and (3) back into (1) to get

$$\begin{aligned} L_1 &= \lim_{x\to 0} \left[\frac{\int_0^x f(t)\,dt}{x} - \frac{\int_0^x tf(t)\,dt}{x^2}\right] \\ &= \lim_{x\to 0} \left[\frac{\int_0^x f(t)\,dt}{x}\right] - \lim_{x\to 0} \left[\frac{\int_0^x tf(t)\,dt}{x^2}\right] = f(0) - \frac{f(0)}{2} = \frac{f(0)}{2} \end{aligned}$$