# MAT 137Y - Practice problems
# Unit 9 - Integration methods

1. Warm up! Compute the following integrals:

   (a) $\int e^{-5x+1}\,dx$. *Hint:* Try the substitution $u = -5x + 1$.

   (b) $\int x^4 \sin\left(2x^5 + 1\right)\,dx$. *Hint:* Try a substitution.

   (c) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\,d\theta$. *Hint:* Try a substitution.

   (d) $\int_1^2 \sqrt{x-1}\,(x+1)\,dx$. *Hint:* Try the substitution $u = x - 1$.

   (e) $\int x\sin(3x)\,dx$. *Hint:* Try integration by parts with $u = x$ and $dv = \sin(3x)dx$.

   (f) $\int_1^2 x^3 \ln x\,dx$. *Hint:* Try integration by parts.

2. Compute the following integrals:

   (a) $\int x^5 e^x\,dx$

   (b) $\int x^2\sqrt{2+x}\,dx$

   (c) $\int_{1/4}^{1/2} \frac{1}{x\ln x}\,dx$

   (d) $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

   (e) $\int \cos\sqrt{t}\,dt$

   (f) $\int x^2 \arcsin x^3\,dx$

   (g) $\int x\arctan x\,dx$

   (h) $\int \sin^5 x\cos^3 x\,dx$

   (i) $\int \arctan\sqrt{x}\,dx$

   (j) $\int_0^{\pi/2} \sin^4 x\,dx$

   (k) $\int \frac{2x+3}{x^2-7x+10}\,dx$

   (l) $\int \frac{x^3}{(x+1)^2}\,dx$

   (m) $\int \frac{x^2-2x}{x^3-3x^2+7}\,dx$

   (n) $\int \frac{2x+3}{x^2+1}\,dx$

   (o) $\int \frac{x}{2x^2-x+2}\,dx$

   (p) $\int \frac{dx}{\sin x}$

   (q) $\int \sec^4 x\tan^4 x\,dx$

   (r) $\int xe^x\cos x\,dx$

3. For each non-negative integer $n$, we define $I_n = \displaystyle\int_0^{2\pi} \sin^n x \, dx$.

   (a) Calculate $I_0$ and $I_1$.

   (b) Starting with $I_n$ use integration by parts. Then use the main trig identity to obtain an equation that relates $I_n$ and $I_{n-2}$.

   (c) Use the previous two questions to get a formula for $I_n$ for all $n$.

4. Consider the following function:

$$E(x) = \int_0^x e^{-t^2} dt.$$

   With a minor variation, this function appears very often in statistics and it is *tabulated.* This means that the numerical values of this function are stored in computers and spreadsheets (and in the old times on tables on books), just like the numerical values of sin or ln are.

   Write the following quantities in terms of $E$:

   (a) $\displaystyle\int_1^2 e^{-t^2} dt$

   (b) $\displaystyle\int_0^x t^2 e^{-t^2} dt$

   (c) $\displaystyle\int_0^x e^{-2t^2} dt$

   (d) $\displaystyle\int_0^1 e^{-t^2+6t} dt$

   (e) $\displaystyle\int_{x_1}^{x_2} e^{-\frac{(t-\mu)^2}{\sigma^2}} dt$

   (f) $\displaystyle\int_0^x \frac{e^{-t}}{\sqrt{t}} dt$

## Some answers and hints

1. (a) Substitution $u = -5x + 1$

   Answer: $\displaystyle\int e^{-5x+1} dx = -\frac{1}{5}e^{-5x+1} + C$

   (b) Substitution $u = 2x^5 + 1$

   Answer: $\displaystyle\int x^4 \sin\left(2x^5+1\right)\, dx = \frac{-1}{10}\cos\left(2x^5+1\right) + C$

   (c) Substitution $u = \cos\theta$

   Answer: $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta}\, d\theta = \frac{1}{2}$

   (d) Substitution: $u = x - 1$

   Answer: $\displaystyle\int_1^2 \sqrt{x-1}\,(x+1)\, dx = \frac{26}{15}$

   (e) Integration by parts.

   Answer: $\displaystyle\int x\sin(3x)\, dx = -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C$

   (f) Integration by parts.

   Answer: $\displaystyle\int_1^2 x^3 \ln x\, dx = \left[\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right]_{x=1}^{x=2} = 4\ln 2 - \frac{15}{16}$

2. (a) Integration by parts five times. After doing it two or three times, you will be able to guess the pattern. Guess the final answer and check that it works. Answer:

   $$\int x^5 e^x\, dx = \left(x^5 - 5x^4 + 5\cdot 4x^3 - 5\cdot 4\cdot 3x^2 + 5\cdot 4\cdot 3\cdot 2x - 5!\right)e^x + C$$

   (b) Use substitution $u = 2 + x$

   Answer: $\displaystyle\int x^2\sqrt{2+x}\, dx = \frac{2}{7}(2+x)^{7/2} - \frac{8}{5}(2+x)^{5/2} + \frac{8}{3}(2+x)^{3/2} + C$

   (c) Use substitution $u = \ln x$. Beware of signs and absolute values.

   Answer: $\displaystyle\int_{1/4}^{1/2} \frac{1}{x\ln x}\, dx = \ln\left|\ln\frac{1}{2}\right| - \ln\left|\ln\frac{1}{4}\right| = -\ln 2$

   (d) Use substitution $x = \sqrt{t}$

   Answer: $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt = 2\sin\sqrt{t} + C$

(e) Use substitution $x = \sqrt{t}$. Then do integration by parts.

Answer: $\displaystyle\int \cos\sqrt{t}\, dt = 2\sqrt{t}\sin\sqrt{t} + 2\cos\sqrt{t} + C$

(f) First substitution $t = x^3$, then integration by parts.

Answer: $\displaystyle\int x^2 \arcsin x^3 \, dx = \frac{1}{3}x^3 \arcsin x^3 + \frac{1}{3}\sqrt{1-x^6} + C$

(g) First integration by parts, then do long division in the fraction.

Answer: $\displaystyle\int x \arctan x \, dx = \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C$

(h) Both $u = \sin x$ or $u = \cos x$ work. The second is easier. Beware as the final answer may look different depending on the substitution you use.

Answer: $\displaystyle\int \sin^5 x \cos^3 x \, dx = \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C$

(i) Use substitution $u = \sqrt{x}$ and integration by parts, in any order.

Answer: $\displaystyle\int \arctan\sqrt{x}\, dx = (x+1)\arctan\sqrt{x} - \sqrt{x} + C$

(j) Half-angle formulas twice.

Answer: $\displaystyle\int_0^{\pi/2} \sin^4 x \, dx = \frac{3\pi}{16}$

(k) First decompose: $\displaystyle\frac{2x+3}{x^2-7x+10} = \frac{13}{3(x-5)} - \frac{7}{3(x-2)}$.

Answer: $\displaystyle\int \frac{2x+3}{x^2-7x+10}\, dx = \frac{13}{3}\ln|x-5| - \frac{7}{3}\ln|x-2| + C$

(l) You could do long division, but it is perhaps easier to use the substitution $u = x+1$.

Answer: $\displaystyle\int \frac{x^3}{(x+1)^2}\, dx = \frac{1}{2}x^2 - 2x + \frac{1}{x+1} + 3\ln|x+1| + C$

(m) This is the easiest question on the list! Substitution $u = x^3 - 3x^2 + 7$.

Answer: $\displaystyle\int \frac{x^2-2x}{x^3-3x^2+7}\, dx = \frac{1}{3}\ln\left|x^3-3x^2+7\right| + C$

(n) Just separate as the sum of two integrals.

Answer: $\displaystyle\int \frac{2x+3}{x^2+1}\, dx = \ln(x^2+1) + 3\arctan x + C$

(o) First complete the square $2x^2 - x + 2 = 2\left(x - \frac{1}{4}\right)^2 + \frac{15}{8}$. Then use the substitution $u = x - 1/4$. Then repeat what you did in Question (2n).

Answer: $\displaystyle\int \frac{x}{2x^2 - x + 2}\,dx = \frac{1}{4}\ln(2x^2 - x + 2) + \frac{1}{2\sqrt{15}}\arctan\frac{4x-1}{\sqrt{15}} + C$

(p) Substitution $u = \cos x$, then partial-fraction decomposition.

$$\int \frac{dx}{\sin x} = \int \frac{-\sin x\,dx}{-\sin^2 x} = \int \frac{du}{u^2 - 1} = \int \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] du = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

(q) Substitution $u = \tan x$.

Answer: $\displaystyle\int \sec^4 x \tan^4 x\,dx = \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$

(r) There are at least four different ways to solve this using integration by parts. They require doing parts more than once and being very organized. Another possibility is to guess that the final answer must be of the form

$$Axe^x \sin x + Bxe^x \cos x + Ce^x \sin x + De^x \cos x,$$

take derivatives, match up the coefficients, and solve the linear equations.

Answer: $\displaystyle\int xe^x \cos x\,dx \;=\; \frac{1}{2}\left[xe^x(\sin x + \cos x) - e^x \sin x\right] + C$

3. To verify your answer, check that $I_8 = \dfrac{35}{64}\pi$.

4. (a) $E(2) - E(1)$

(b) $\dfrac{1}{2}\left[E(x) - xe^{-x^2}\right]$

(c) $\dfrac{E(\sqrt{2}x)}{\sqrt{2}}$

(d) Complete the square. $e^9\left[E(-2) - E(-3)\right]$

(e) $\sigma E\left(\dfrac{x_2 - \mu}{\sigma}\right) - \sigma E\left(\dfrac{x_1 - \mu}{\sigma}\right)$

(f) For $x > 0$, $2E(\sqrt{x})$