MAT 137Y - Practice problems
Unit 12 - Improper integrals

1. Determine whether the following integrals are convergent or divergent directly from the definition of improper integral. If they are convergent, calculate their value.

   (a) $\displaystyle\int_{-1}^{\infty} \frac{1}{x^2+1}\,dx$

   (b) $\displaystyle\int_{0}^{1} \ln x\,dx$

   (c) $\displaystyle\int_{0}^{\infty} \cos x\,dx$

   (d) $\displaystyle\int_{0}^{1} \frac{dx}{x^2}$

   (e) $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x}}$

   (f) $\displaystyle\int_{2}^{\infty} \frac{1}{x^2-1}\,dx$

   Hint: For Question (1f), write $\displaystyle\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$.

2. (a) For which values of $p > 0$ is the integral $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ convergent?

   (b) For which values of $p > 0$ is the integral $\displaystyle\int_{0}^{1} \frac{1}{x^p}\,dx$ convergent?

   (c) Let $a, b \in \mathbb{R}$. Assume $a < b$.
   For which values of $p > 0$ is the integral $\displaystyle\int_{a}^{b} \frac{1}{(x-a)^p}\,dx$ convergent?

3. Using the Basic Comparison Test and/or the Limit-Comparison Test, determine which ones of the following improper integrals are convergent or divergent. Do not calculate their value.

   (a) $\displaystyle\int_{1}^{\infty} \frac{\sin x + 2\cos x + 10}{x^2}\,dx$

   (b) $\displaystyle\int_{0}^{\infty} \frac{x-7}{x^2+x+5}\,dx$

   (c) $\displaystyle\int_{10}^{\infty} \frac{\sqrt{x-6}}{3x^2+5x+11}\,dx$

   (d) $\displaystyle\int_{0}^{\infty} \frac{\arctan x}{x^{1.1}}\,dx$

   (e) $\displaystyle\int_{0}^{1} \frac{\sin x}{x^{4/3}}\,dx$

   (f) $\displaystyle\int_{0}^{\infty} e^{-x^2}\,dx$

4. Let $a < b$. Let f be a continuous function on $[a, \infty)$. Prove that the following two statements are equivalent:

   - The improper integral $\displaystyle\int_{a}^{\infty} f(x)dx$ is convergent.
   - The improper integral $\displaystyle\int_{b}^{\infty} f(x)dx$ is convergent.

   Write a formal proof directly from the definition of improper integral as a limit.

   Suggestion: Use the limit laws. You do not need to get dirty with epsilons.

5. Review the statement of the Limit-Comparison Test (Video 12.9). There are two generalizations of the theorem.

   (a) Assume the limit L in the theorem exists and is 0. The full conclusion of the theorem is no longer true but we can still draw some conclusions in some cases. If one of $\int_a^{\infty} f(x)dx$ or $\int_a^{\infty} g(x)dx$ is convergent or divergent, can we conclude something about the other?

   Figure out what the correct conclusions are, and write a proof (imitating the proof in Video 12.10).

   (b) Repeat the same question when the limit L is ∞.

6. For which values of $a, b \in \mathbb{R}$ are each of the following improper integrals convergent or divergent?

   (a) $\displaystyle\int_2^{\infty} \frac{1}{x^a (\ln x)^b}\, dx$ (b) $\displaystyle\int_1^{2} \frac{1}{x^a (\ln x)^b}\, dx$ (c) $\displaystyle\int_1^{\infty} \frac{1}{x^a (\ln x)^b}\, dx$

   Note: This is a long question. You will have to break each integral into cases (depending on values of a and b). You will likely use BCT, LCT, the definition of improper integral, and substitution at different points. For Question 6b, we suggest studying the case $a = 1$ first.

7.
   - A type-1 improper integral is an integral of the form $\int_c^{\infty} f(x)dx$, where f is a continuous, bounded function on $[c, \infty)$.
   - A type-2 improper integral is an integral of the form $\int_a^{b} f(x)dx$, where f is a continuous function on $(a, b]$ (possibly with vertical asymptote $x = a$) or on $[a, b)$ (possibly with vertical asymptote $x = b$).

   In the Videos we explicitly wrote the statement (and proof) for BCT and LCT for type-1 improper integrals, but we have also been using them for type-2 improper integrals. Write the statements and proofs.

Some answers and hints

1. (a) Convergent: $\frac{3\pi}{4}$
   (b) Convergent: -1
   (c) Divergent: oscillating
   (d) Divergent: ∞
   (e) Convergent: 2
   (f) Convergent: $\frac{1}{2}\ln 3$

2. (a) Convergent iff $p > 1$
   (b) Convergent iff $p < 1$
   (c) Convergent iff $p < 1$

3. (a) Convergent
   (b) Divergent
   (c) Convergent
   (d) Convergent
   (e) Convergent
   (f) Convergent

4.

$$\int_a^\infty f(x)dx = \lim_{x\to\infty} F(x) \quad \text{where} \quad F(x) = \int_a^x f(t)dt$$
$$\int_b^\infty f(x)dx = \lim_{x\to\infty} G(x) \quad \text{where} \quad G(x) = \int_b^x f(t)dt$$

Notice that $G(x) = F(x) + M$ where M is a fixed number (which number?) Use limit law for sum of functions.

5. Let us call $P = \int_a^\infty f(x)dx$ and $Q = \int_a^\infty g(x)dx$. Let $L = \lim_{x\to\infty} \frac{f(x)}{g(x)}$.

   (a) Assume $L = 0$.
   - If $P = \infty$ then $Q = \infty$.
   - If $Q < \infty$ then $P < \infty$

   (b) Assume $L = \infty$.
   - If $Q = \infty$ then $P = \infty$.
   - If $P < \infty$ then $Q < \infty$.

   We cannot draw any other conclusions.

6. (a) Convergent when $a > 1$. Convergent when $a = 1$ and $b > 1$. Divergent otherwise.
   (b) Convergent when $b < 1$. Divergent when $b \geq 1$. The value of a does not matter.
   (c) Convergent when $a > 1$ and $b < 1$. Divergent otherwise.