VOTING THEORY

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Canada/USA Mathcamp 2015

An updated copy of these notes is available online at
`http://math.toronto.edu/~alfonso/MC.html`

1 Single-winner voting systems

1.1 A warm-up exercise

We are running an election with 5 candidates. You have access to a very precise poll that gives you the full preferences of each voter:

```
31: A > D > C > E > B
20: B > D > C > E > A
19: D > C > E > B > A
16: E > C > B > A > D
14: C > E > D > B > A
```

This means that there are 31 voters who have A as their top choice, D as their second choice, then C, then E, and finally B as their least favourite candidate. And so on.

Come up with 5 voting methods which all sound fair, and such that each one of them will make a different candidate win.

Don't read ahead yet if you do not want to be spoiled!

1.2 Voting systems and voting criteria - the short list

- COMMON VOTING SYSTEMS
  - **Plurality:** Each voter votes for their preferred candidate. The candidate with the most votes wins.
  - **Top-two run-off:** First, each voter votes for their preferred candidate. We select the two candidates with the most votes, and we have a second round only with them. The candidate among the two with the most votes wins.
  - **Borda Count:** If there are N candidates, each voter assigns $N-1$ points to their preferred candidate, $N-2$ votes to their second preferred candidate, and so on, to 0 points for their least-preferred candidate. The candidate with the most points wins.
  - **Instant Run-off Voting:** Each voter votes for their favourite candidate. The candidate with the least votes is eliminated. Repeat until there is only one candidate left.
  - **Survivor:** Each voter votes for their least favourite candidate. The candidate with the most votes is eliminated. Repeat until there is only one candidate left.
- SIMPLE VOTING CRITERIA
  - **Majority criterion:** If a majority (i.e. more than 50%) of the voters have X as top choice, then X wins.
  - **Majority–loser criterion:** If a majority of voters have X as their last choice, then X does not win.
  - **Pareto Criterion:** If every voter prefers X to Y, then Y does not win.
  - **Condorcet criterion:** Assume there is a candidate X such that, for any other candidate Y, X would win against Y in a head-to-head race just between the two of them. Then X wins.

Exercise: For each of the five voting systems above, which ones of the four criteria do they pass or do they fail? For example, Plurality passes majority criterion. On the other hand, Survivor fails the Majority Criterion because we can construct a scenario in which a candidate with more than 50% of the votes loses the election.

1.3 Voting systems and voting criteria - the long list

- VOTING SYSTEMS
  - **Plurality:** Each voter votes for their preferred candidate. The candidate with the most votes wins.
  - **Top-two run-off:** First, each voter votes for their preferred candidate. We select the two candidates with the most votes, and we have a second round only with them. The candidate among the two with the most votes wins.
  - **Borda Count:** If there are N candidates, each voter assigns $N-1$ points to their preferred candidate, $N-2$ votes to their second preferred candidate, and so on, to 0 points for their least-preferred candidate. The candidate with the most points wins.
  - **Instant Run-off Voting:** Each voter votes for their favourite candidate. The candidate with the least votes is eliminated. Repeat until there is only one candidate left.
  - **Survivor:** Each voter votes for their least favourite candidate. The candidate with the most votes is eliminated. Repeat until there is only one candidate left.
  - **Approval:** Each voter votes for as many candidates as they want, giving one point to each. The candidate with the most points wins.
  - **Random ballot:** Each voter votes for their preferred candidate. We put all the ballots in a big hat and we pick one at random. That is the winner.
- VOTING CRITERIA
  - **Majority criterion:** If a majority (i.e. more than 50%) of the voters have X as top choice, then X wins.
  - **Majority–loser criterion:** If a majority of voters have X as their last choice, then X does not win.
  - **Pareto Criterion:** If every voter prefers X to Y, then Y does not win.
  - **Condorcet criterion:** Assume there is a candidate X such that, for any other candidate Y, X would win against Y in a head-to-head race just between the two of them. Then X wins.
  - **Condorcet-loser criterion:** Assume there is a candidate X such that, for any other candidate Y, X would lose against Y in a head-to-head race just between the two of them. Then X does not win.
  - **Consistency criterion:** If we have two sets of ballots such that X would win on each one of them, then X would also win on the union of the two sets counted together.
  - **Monotonicity criterion:** If we start in a situation where X wins and we modify the voters' preferences by increasing the position of X in some of them, then X still wins.
  - **Participation criterion:** Let X and Y be two candidates. If we start in a situation where X wins, and we add some new ballots that all rank X above Y, then Y does not win.
  - **Independence of clones:** Let S be a set of choices such that no voter ranks a choice not in S between two choices in S. Then removing some (but not all) of the options in S has no effect on whether the winner is a choice in S or not.

– **Independence of Irrelevant Alternatives:** Let X and Y be two candidates. If we start in a situation where X wins and some voters change their preferences, but nobody changes their relative preference between X and Y, then Y does not win.

– **Invulnerability to strategy:** A voter or a group of voters can never benefit from voting dishonestly.

1.4 Exercises

1. You already filled in the top four rows entries on this table. Complete the shaded entries: figure out whether the corresponding voting system passes or fail the corresponding voting criterion. If it fails, you need an example. If it passes, you need a proof or an argument.

   There is an answer in the next page.

	Plurality	Top 2	Borda	IRV	Survivor
Majority	PASS	PASS	FAIL	PASS	FAIL
Majority-loser	FAIL	PASS	PASS	PASS	PASS
Pareto	PASS	PASS	PASS	PASS	PASS
Condorcet	FAIL	FAIL	FAIL	FAIL	FAIL
Condorcet-loser					
Consistency					
Participation					
Monotonicity					
Independency of clones					
IIA					

2. Come up with a voting method that passes Condorcet.

3. Come up with a voting method that passes IIA.

4. **Impossible challenge:** Come up with a voting method that passes Condorcet and IIA.

1.5 Partially filled-out table

	Plurality	Top 2	Borda	IRV	Survivor
Majority	PASS	PASS	FAIL (a)	PASS	FAIL (a)
Majority-loser	FAIL (b)	PASS	PASS (c)	PASS	PASS
Pareto	PASS	PASS	PASS	PASS	PASS
Condorcet	FAIL (d)	FAIL (d)	FAIL (d)	FAIL (d)	FAIL (a)
Condorcet-loser	FAIL (j)	PASS		PASS	
Consistency	PASS	FAIL (f)	PASS	FAIL (f)	FAIL (f)
Participation	PASS	PASS	PASS		
Monotonicity	PASS	FAIL (e)	PASS	FAIL (e)	
Independency of clones	FAIL (g)	FAIL (g)	FAIL (h)	PASS	FAIL (g,h)
IIA	FAIL (i)	FAIL (i)		FAIL (i)	

References from the table

(a) A is the top choice of a majority of voters. A is also a Condorcet winner. However, B wins with Borda and with Survivor.

```
3: A > B > C > D
2: A > B > D > C
4: B > C > D > A
```

(b) A is the last choice of a majority of voters, but A wins with Plurality.

```
3: B > C > A
2: C > B > A
4: A > B > C
```

(c) Use pigeonhole principle.

(d) D is a Condorcet winner, but D loses with Plurality, Top-2, Borda, and IRV.

```
31: A > D > C > E > B
20: B > D > C > E > A
19: D > C > E > B > A
16: E > C > B > A > D
14: C > E > D > B > A
```

(e) In the scenario below, A wins with Top-2 and with IRV. However, if exactly exactly 3 voters changed from $C > A > B$ to $A > C > B$, thus improving their ranking of A, then B wins.

```
3: C > A > B
10: A > C > B
8: B > A > C
6: C > B > A
```

(f) Below are two scenarios with candidates A, B, C, D. Each of the scenarios would be won separately by A with Top-2, with IRV, and with Survivor. The union of both scenarios would be won by B with Top-2, by B with IRV, and by D with Survivor.

```
Scenario 1:
        4: A > C > D > B
        3: C > B > D > A
        2: B > A > C > D
Scenario 2:
        5: A > D > B > C
        4: D > B > C > A
        3: B > A  > C > D
```

(g) In the scenario below C, D, and E are clones. When they all run, A wins with Plurality, Top-2, and Survivor. If two of the clones drop out of the race, then the third clone wins.

```
24: A > B > C > D > E
23: B > A > C > D > E
22: C > D > E > A > B
21: D > E > C > A > B
10: E > D > C > B > A
```

(h) In the scenario below A and B are clones. When they all run, A wins with Survivor and with Borda. If one clone drops out of the race, then C wins.

```
4: A > B > C
3: C > A > B
2: C > B > A
```

(i) In Scenario 1, A wins with Plurality, Top-2, and IRV. To go from Scenario 1 to Scenario 2, exactly 5 voters changed from $A > B > C$ to $C > A > B$, and exactly 1 voter changed from $A > B > C$ to $A > C > B$. Hence, nobody changed their relative order between A and B. However, in Scenario 2, B wins.

```
Scenario 1:
        9: A > B > C
        6: B > A > C
Scenario 2:
        3: A > B > C
        1: A > C > B
        5: C > A > B
        6: B > A > C
```

(j) A is a Condorcet loser but A wins with Plurality.

```
4: A > B > C
3: B > C > A
2: C > B > A
```

2 Condorcet methods

All of the more common methods fail the Condorcet criterion. Here are a few methods that pass the criterion.

All these methods start by building the *graph of voter preferences*. For each pair of distinct candidates A and B, let us call $f(A,B)$ the number of voters that prefer A to B. Next, build a graph with one vertex per candidate. Given a pair of vertices A and B, if $f(A,B) \geq f(B,A)$ then we draw a (directed) edge from A to B with weight $f(A,B)$. Notice that given every pair of vertices, we will have an edge from the first to the second or from the second to the first. (We assume ties never happen, which is true in practice for large numbers of voters.)

Notice that A is a Condorcet winner iff there no incoming arrows at vertex A.

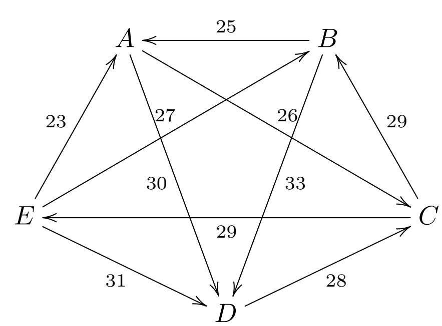
- **Example:** (Stolen from wikipedia.) Assume we have 45 voters and 5 candidates. Assume we have the following voter preferences:

```
5: A > C > B > E > D
5: A > D > E > C > B
8: B > E > D > A > C
3: C > A > B > E > D
7: C > A > E > B > D
2: C > B > A > D > E
7: D > C > E > B > A
8: E > B > A > D > C
```

  In this example

  - $5+5+3+7 = 20$ voters prefer A to B
  - 25 voters prefer B to A.

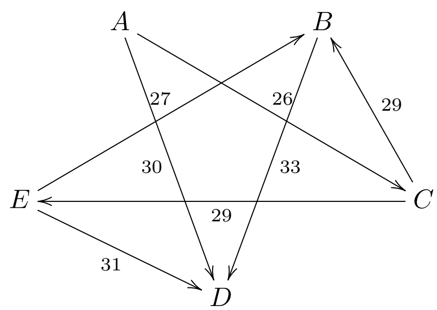
  Hence $f(A,B) = 20$ and $f(B,A) = 25$. So, in the graph of voter preferences, we will draw an arrow from B to A and label it 25. Here is the full graph of voter preferences:



2.1 Ranked pairs

Consider the graph of voter preferences. We are going to look at all the edges in decreasing order of weight; we will decide in that order whether we "lock" each edge. At each step, we lock a particular edge iff it would not produce a cycle among the already locked edges, and otherwise we simple move to the next edge. When we are done, the edges that have been locked will produce a total order among the candidates that we use to pick our winner.

In our example above, this procedure will make us lock all the edges except the ones with weight 28, 25, 23, producing the graph:



Hence, the outcome of this election is $A > C > E > B > D$.

2.2 Schultze

Schultze is a method designed specifically so that it will satisfy a lot of desirable criteria which are not often satisfied, at the expense of being mathematically complex. Here is the procedure:

1. We start by computing the function f described above and by drawing the graph of voter preferences described above.

2. Consider a pair of candidates X and Y. We assign to each oriented path from X to Y in the graph the weight of its weakest link. Then, we define $p(X, Y)$ to be the largest weight among all the paths from X to Y (and 0 if there is no such path).

   For instance, let us calculate $p(A, B)$ in our example above. There are 4 paths from A to B, shown below. For each one of them, the segment that gives the weight to the path is indicated with a double arrow.

   $$A \xRightarrow{\mathbf{26}} C \xrightarrow{29} B$$

   $$A \xRightarrow{\mathbf{26}} C \xrightarrow{29} E \xrightarrow{27} B$$

   $$A \xrightarrow{30} D \xRightarrow{\mathbf{28}} C \xrightarrow{29} B$$

   $$A \xrightarrow{30} D \xrightarrow{28} C \xrightarrow{29} E \xRightarrow{\mathbf{27}} B$$

   Hence $p(A, B) = \max(26, 26, 28, 27) = 28$.

3. Finally, we say that society prefers X to Y (and we write $X > Y$) if $p(X, Y) > p(Y, X)$.

Theorem 1: This produces a total order in the set of candidates! This is the outcome of an election with the Schultze method.

This theorem is not at all obvious. I leave the proof as an exercise. I hope you now see why I said this method was complex. But it passes a lot of nice criteria!

How to tally a Schultze election in practice.

Assume our set of candidates is $\{A_1, \ldots, A_N\}$. Let us define a matrix M whose (i,j)-th entry is $f(A_i, A_j)$, and whose (i, i)-th entry is the total number of voters. We call this matrix the *matrix of voter preferences*. This is easy to build from the graph of voter preferences. In our example above:

$$M = \begin{bmatrix} 45 & 20 & 26 & 30 & 22 \\ 25 & 45 & 16 & 33 & 18 \\ 19 & 29 & 45 & 17 & 24 \\ 15 & 12 & 28 & 45 & 14 \\ 23 & 27 & 21 & 31 & 45 \end{bmatrix}$$

Now let us define a new $N \times N$ matrix, called the *the matrix of path preferences* L, whose (i, j)-th entry is $p(A_i, A_j)$. This is hard to build. In the example above, after a lot of computations, we get

$$L = \begin{bmatrix} 45 & 28 & 28 & 30 & 24 \\ 25 & 45 & 28 & 33 & 24 \\ 25 & 29 & 45 & 29 & 24 \\ 25 & 28 & 28 & 45 & 24 \\ 25 & 28 & 28 & 31 & 45 \end{bmatrix}$$

Once we have the matrix L, it is very easy to read off the outcome of a Schultze election. For every pair of voters we just have to compare $p(A_i, A_j)$ and $p(A_j, A_i)$. We mark the winner of each two-candidate confrontation in red:

$$L = \begin{bmatrix} 45 & \color{red}{28} & \color{red}{28} & \color{red}{30} & 24 \\ 25 & 45 & 28 & \color{red}{33} & 24 \\ 25 & \color{red}{29} & 45 & \color{red}{29} & 24 \\ 25 & 28 & 28 & 45 & 24 \\ \color{red}{25} & \color{red}{28} & \color{red}{28} & \color{red}{31} & 45 \end{bmatrix}$$

Hence, the Schultze method produces an outcome of $E > A > C > B > D$ in this example.

The "only" difficult step is how to construct L from M. Luckily, we have a cute theorem:

Theorem 2: As matrices, L is the $(N-1)$-st power of M, i.e. $L = M^{N-1}$, if we perform matrix multiplication using min instead of products or real numbers and max instead of sums of real numbers.

Proof. Exercise. It is fun! □

3 Multi-winner voting systems

3.1 Exercises

1. We are holding an election for the Luxembourg parliament. There are five parties and 1000 voters. Each voter votes for their favourite party. We need to elect 5 representatives. Design a model to tally the votes and choose the 5 representatives given the ballots. A desirable property of the voting method is *proportional representation*: If party A gets n times the number of votes that party B gets, then party A should have roughly n times the number of representatives that B receives.

   Apply your method to the following results:

   ```
   Party A: 594 votes
   Party B: 235 votes
   Party C: 100 votes
   Party D: 60 votes
   Party E: 11 votes
   ```

   Repeat with the same results if we need to elect 10 representatives instead of 5, then for 20 representatives instead of 5 or 10.

2.
   (a) We are holding an election for parliament again. Our country has five provinces and we have decided that each province will select its own representatives. The total number of representatives in parliament is 20. We want the number of representatives of each province to be proportional to their population (which are $594,000$, $235,000$, $100,000$, $60,000$, and $11,000$). How many representatives should each province receive?

   (b) Repeat the same problem, but now with the added restriction that each province needs to have at least one representative, while the number of seats per province is still as close to proportional to population as possible.

3.2 D'Hondt and Saint Laguë methods

There are two standard methods to assign parliamentary seats to parties proportional to the number of votes. We define the quota Q to be the number of votes that every seat costs. Then, if a party has V votes, we give the party V/Q seats, rounded down to an integer (D'Hondt method) or rounded to the closest integer (Saint Laguë method).

In both methods, the subtle point is that the quote is chosen so that the total number of seats elected will equal the number of seats in parliament.

There is a fast trick to compute this (basically, to answer questions like those in the exercises in §3.1) using a table, which we explained in class.