

Math of Election – APPORTIONMENT

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Apportionment refers to any of the following, equivalent mathematical problems:

- We want to elect a Congress with a total of N seats. We need to distribute them among various states so that the number of seats per state is (roughly) proportional to its population.
- A country needs to elect N representatives. Instead of dividing the state into N districts, each party presents various candidates for the state as a whole. Each voter votes for a party. Then we assign to each party a number of representatives (roughly) proportional to the number of votes they got.

Warm up: Hamilton method and paradoxes

Let P be the total population of all states combined and let N be the total number of seats. Let $H = \frac{P}{N}$ be the Hare Quota. Ideally, to state i with population P_i we would assign $q_i = \frac{P_i}{H}$. Sadly, this is not an integer. Instead, to state i we assign $\lfloor q_i \rfloor$ seats and keep track of the fractional part. This assigns a total less than N seats. Then we assign the remaining seats, one by one, to the states with the largest fractional part of q_i , until we obtain the right total. This method of apportionment is called *Hamilton* or *Largest Remainder*.

The following problems will show you that this method does not behave very well.

1. Distribute seats among the following states with the Hamilton method:

State	Population
A	1,500
B	1,000
C	100

Do the distribution twice, once for a total of 37 seats, and once for a total of 38 seats. The result should make you uncomfortable. This is called the “*Alabama paradox*”.

2. Distribute 100 seats among the following states with the Hamilton method:

State	Population (2000 census)	Population (2010 census)
A	746	750
B	2,217	2,128
C	7,037	6,960

We have the data from the census at two times, so do the distribution twice. The result should make you uncomfortable. This is called the “*Population paradox*”.

3. Consider a country with the following two states:

State	Population
A	10,300
B	37,700

- (a) Distribute 100 seats among them using the Hamilton method.
- (b) Now assume new state C joins this country, with a population of 3,000. With the apportionment already done and without changing the number of seats for states A and B , how many seats should state C receive?
- (c) Ok, then we are going to increase the size of the house by adding as many seats as your answer to the previous question. Now, just to make sure, redistribute the total number of seats among states A , B , and C using the Hamilton method from scratch.
- (d) Did you observe something fishy? That is the “*New States Paradox*”.

Divisor methods

Here are three methods of apportionment that behave very different from Hamilton. Let's say we want to distribute N seats among states. Let Q be a positive number, called the quota, that we will fix later. Then to each state i with population P_i , we want to assign a number of seats n_i equal to $\frac{P_i}{Q}$ rounded to an integer. The three methods differ in how we round:

- The *Jefferson* method (a.k.a. D'Hondt, Greatest Divisors, or Down): always round down.
- The *Adams* method (a.k.a. Smallest Divisors, or Up): always round up.
- The *Webster* method (a.k.a. Saint-Laguë, Scheppers, Major Fractions, or Arithmetic): round to the closest integer.

Proceeding this way, vary the number Q until you find a number that produces a total number of seats equal to N , as we wanted.

In the following problems you will examine some of their nice properties, learn a more complex method (currently used in the US) and see how they would have changed the outcome of the 2000 US presidential election.

4. Here is a “new” method. We want to distribute N seats. For state i with population P_i , write the sequence of numbers

$$\frac{P_i}{1}, \frac{P_i}{2}, \frac{P_i}{3}, \frac{P_i}{4}, \frac{P_i}{5}, \dots$$

Look at all the sequences of numbers you have written. Select the largest N numbers. Give to each state as many seats as numbers you have chosen in its sequence.

This method is not new. It is equal to one of the three methods described above. Figure out which one and why.

5. We have given you the description of three methods as “rounding methods” and the description of one of them also as a “divisor method”. Come up with a description of the other two methods as divisor methods.
6. Prove that a divisor method can never suffer from the Alabama Paradox, the Population Paradox, or the New States Paradox.
7. There could be other ways of “rounding”. Each one of them would produce a new divisor method. Describe the most general rounding/divisor method that you can think of.
8. The apportionment method currently used in the US was derived in 1920 by Huntington and Hill as a method with a certain mathematical property called “stability”. We will tell you what stability means, and you will figure out how the method works! If you want to see the original paper, the reference is
 - Huntington, E.V.: *The mathematical theory of the apportionment of representatives*, Proc Natl Acad Sci USA, 1921 Apr; 7(4): pages 123–127

Huntington says:

“... between any two states there will practically always be a certain inequality which gives one of the states a slight advantage over the other. A transfer of one representative from the more favoured state to the less favoured state will ordinarily reverse the sign of this inequality, so that the more favoured state now becomes the less favoured, and vice versa. Whether such a transfer should be made or not depends on whether the amount of inequality between the two... is less or greater than it was before...”

If a state has population P and it receives n seats, then the *district size* of that state is $r = P/n$. If two states have district sizes r_i and r_k with $r_i > r_k$, then state k is being favoured over state i . In an ideal situation $r_i = r_k$ always, but of course that is not possible. Huntington proposes, as a measure of inequality, the quantity

$$T = \frac{r_i}{r_k} - 1$$

Notice that we always divide the *larger* district size over the *smaller* district size. A distribution of seats is stable when a transfer of seats between one state and another would never decrease the value of T for that pair of states.

Now assume we are in a stable distribution. Assume state k is favoured over state i but if we transfer one seat from k to i the quantity T would increase. Write down this statement as an inequality (careful: which state would be favoured after the transfer?) Manipulate the inequality so that you have all quantities about i on one side and all quantities about k on the other side. Try to come up with an apportionment method in terms of divisors that will be stable. (*Hint*: Think of divisors.)

9. We are going to see why choosing different methods of apportionment matter by looking at the closest presidential election in US history: George Bush vs Al Gore in 2000. You are going to apply three apportionment methods to the results of this election: Hamilton, Hill, and Jefferson. You will get three very different results! Notice that each of these methods has actually been used for decades in the US. We will proceed in steps.

- **The House of Representatives or Congress.** The US Congress has 435 seats. They are distributed among 50 states proportionally to their population, but with the constraint that each state should get at least 1. We need to use the 1990 population census for the 2000 election. You can download it from <http://tinyurl.com/apportionment2017>
Distribute the seats among the states in three different ways: using Hamilton, Hill, and Jefferson methods.
- **The Senate.** The US Senate has 100 seats, exactly two seats for each state.
- **The Electoral College.** The US Electoral College has 538 seats: each state gets the sum of the seats it has in Congress and the seats it has in the Senate. In addition DC receives a total of 3 seats. Calculate the number of seats in Electoral College that each state will have with each of the three methods: Hamilton, Hill, and Jefferson.
- **The President.** Oversimplifying, in a presidential election the candidate that wins on each state receives *all* the electoral college seats for that state. The candidate with the most electoral college seats becomes president. (Actually, there are 2 states that distribute their electoral college seats among both candidates, but in the 2000 election one single candidates won all the seats for each state anyway, so we will leave it at that.) You can download who won each state in the 2000 presidential election from the same url above. Now compute who would have become president in 2000 under Hamilton, Hill, and Jefferson’s methods.

Additional problems

10. Construct an example where the four divisor methods you have learned produce all different outcomes.
11. The four divisor methods have a “bias” in favour of large states or in favour of smaller states. With the previous example in mind, which method do you think favours large or small states? In general, given two methods described by divisors, which one benefits which states?
12. Balinski and Young stated and proved many theorems about apportionment in
 - Balinski, M. and Young, P.: *Fair Representation: Meeting the Ideal of One Man, One Vote*, New Haven and London Yale University Press, 1982, ISBN: 0-300-02724-9. Appendix A: The Theory of Apportionment, pages 95-156.

Probably their most quoted result is their impossibility theorem. It says that it is impossible to have an apportionment method that satisfies the following three properties at once:

- (a) It is symmetric. This means that if you permute the population of states, then their numbers of seats get equally permuted. This simply means that all states are treated equally.
- (b) It does not suffer from the Alabama paradox. This means that if state A 's population increases at a rate faster than state B 's, then it is impossible for state A to lose a seat and for state B to win a seat at the same time. Notice that any divisor method will be fine for this, but not the Hamilton method.
- (c) It stays within quota. If the total population is P , the total number of seats is N , and the population of state i is P_i , then the quota of state i is $q_i = \frac{P_i}{P/N}$. A method stays within quota when the number of seats of a state is always its quota, rounded up or down. Notice that the Hamilton method was specifically designed to satisfy this.

The proof that Balinski and Young present proceeds by contradiction. Assume there is a method with these three properties, and use it to distribute 7 seats among four states in the following two situations:

State	Population (1st Census)	Population (2nd Census)
A	$5 + \varepsilon$	$4 - \varepsilon$
B	$2/3$	$2 - \varepsilon/2$
C	$2/3$	$1/2 + \varepsilon/2$
D	$2/3 - \varepsilon$	$1/2 + \varepsilon$

where ε is a very small number. Then reason why there is a contradiction.

Fill in the details of this proof.